De Bruijn's Weak Diamond Property Revisited

Jörg Endrullis[a], Jan Willem Klop[a,b]

[a] *VU University Amsterdam, The Netherlands*
[b] *Centrum Wiskunde & Informatica (CWI), Amsterdam, The Netherlands*

Abstract

In this paper we revisit an unpublished but influential technical report from 1978 by N.G. de Bruijn, written in the framework of the Automath project. This report describes a technique for proving confluence of abstract reduction systems, called the *weak diamond property*. It paved the way for the powerful technique developed by Van Oostrom to prove confluence of abstract reduction systems, called *decreasing diagrams*.

We first revisit in detail De Bruijn's old proof, providing a few corrections and hints for understanding. We find that this original criterion and proof technique are still worthwhile. Next, we establish that De Bruijn's confluence criterion can be used to derive the decreasing diagrams theorem (the reverse was already known). We also provide a short proof of decreasing diagrams in the spirit of De Bruijn. We finally address the issue of completeness of this method.

1. Introduction

In this article we highlight an important contribution of Dick de Bruijn to the theory of term rewriting systems, as recorded in the technical report '*A Note on Weak Diamond Properties*', from 1978. This note was very influential as we will describe, but it was never published. Yet the original proof of De Bruijn's theorem is still worthwhile, and we will render it here in all details but with some innocent corrections. The proof is worthwhile not only for historical reasons, but also because the style of the proof, using a complicated induction, makes it interesting for mechanical verification.

Remarkably, it was precisely the goal of mechanical verification of mathematical theorems that gave rise to the note of de Bruijn that we will treat. As is well-known, De Bruijn was one of the acknowledged pioneers of mechanical verification of mathematical theorems, with his ground-breaking project Automath that ran from 1967 to the early eighties. See Geuvers, Nederpelt, de Vrijer [13] for an annotated collection of the main papers originated by the

Email addresses: `j.endrullis@vu.nl` (Jörg Endrullis), `j.w.klop@vu.nl` (Jan Willem Klop)

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Automath project. The inheritage of technical reports resulting from the Automath project can be found at the archive `http://www.win.tue.nl/automath/`. For a shorter impression including an appraisal of related endeavours see the paper by Geuvers and Nederpelt in the present issue of this journal.

De Bruijn and his Automath team developed a considerable body of theory about typed lambda calculi, and De Bruijn is generally acknowledged as the co-discoverer of the famous Curry-Howard-De Bruijn principle, often called 'Formulas as types'. For the Curry-Howard-De Bruijn principle see the book [14] by Sørensen and Urzyczyn; for an encompassing up-to-date treatment of typed lambda calculi we refer to [2] by Barendregt, Dekkers and Statman. The standard treatise on pure lambda calculi is Barendregt [1]. In the course of this theory development De Bruijn made several fundamental contributions that have become standard later on. A typical case is how to treat bound variables in lambda calculus, where the concerns of practical implementation led De Bruijn to his widely known and applied 'lambda calculus with nameless dummies'. One of the basic requirements for the various lambda calculi designed in the Automath project is *confluence*, defined in the next section. There are several fundamental abstract lemma's to prove confluence, of which the most famous is the Lemma of Newman [12] (M.H.A. Newman was a well-known topologist and the PhD supervisor of A. Turing). Another well-known lemma is the strong confluence Lemma of Huet [8]. Contemplating such abstract confluence lemma's, De Bruijn discovered in 1977 with his great combinatorial insight a very powerful theorem that had many abstract lemma's about confluence as corollaries, including the two just mentioned.

In March 1990, the second author gave an introductory talk on term rewriting in a seminar at Philips Research Laboratories in Waalre, with Dick de Bruijn present among the audience. A few days afterwards, De Bruijn sent the technical memorandum 78–08 to which the present paper is devoted, with in the accompanying letter the question whether this note, which had remained then dormant for twelve years, possibly contained something of any value. The second author was happy to confirm this in strong terms, stating in his answering letter of March 5, 1990 that this observation was stronger than many of the familiar abstract confluence lemma's. De Bruijn's proof consisted of an ingenious but complicated nested induction, that at first sight looked somewhat forbidding. An attempt for an alternative proof led Klop and de Vrijer in 1990 to a geometrical proof, which much later together with Van Oostrom was published as [11]. The disadvantage of that geometrical proof is the use of analytical and metrical notions such as condensation points in the euclidean plane, notions that are not easily formalised and are not of a finitary combinatorial nature. Another issue of De Bruijn's theorem was that it had a certain asymmetry, as we will see below. Van Oostrom responded early nineties in his dissertation studies ([17, 16]) with a deep analysis of De Bruijn's idea; he was able to give a very different proof using a convenient calculus of multisets, together with establishing as a key notion a certain invariant that he called 'decreasing diagrams'. This not only gave a more elegant symmetrical version of De Bruijn's theorem, in fact extending it to partial orders, but also provided the basis for several useful related extensions [16, 18, 6].

Recently, Van Oostrom found another elegant proof of his decreasing diagram theorem by means of recursive path orders (RPO) and extended to confluence modulo together with Felgenhauer [6], but that will not be covered in the present paper. Let it suffice to mention that the interest of that recent development is that it combines one of the most powerful techniques to prove termination (also known as 'strong normalisation', SN) of first order term rewriting systems, to wit the RPO technique, with the most powerful method to prove confluence for abstract reduction systems, to wit the "weak diamond property annex the decreasing diagram method" of De Bruijn and Van Oostrom, respectively.

Contribution. These historical recollections bring us to formulating the purpose of the present article, apart from presenting the historical record of events itself. Figure 1 gives an overview over this paper.

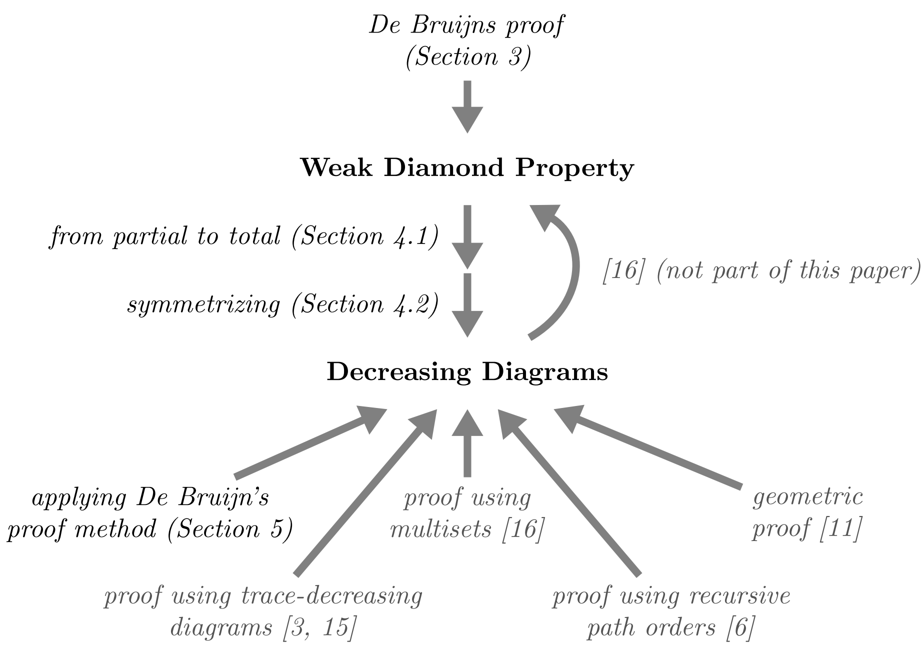


Figure 1: Overview over the paper, and various routes to the De Bruijn's weak diamond property and Van Oostrom's decreasing diagrams. Greyed out parts are not covered in the current paper.

First of all, we present De Bruijn's original proof, in a slightly optimised version for better readability, thereby correcting a few small points. Also several hints and elaborations are given for easier understanding of the proof. Second, we discuss some extensions of De Bruijn's original formulation, which was given for a total order on the set of labels or indexes, to *partially ordered index sets.*

In fact an extension to partially ordered index sets was already mentioned to De Bruijn in the afore-mentioned letter by the second author.

Another matter is how to *symmetrize* the original version of De Bruijn, thereby obtaining Van Oostrom's symmetrical decreasing diagram version. We will show that the symmetrical version can be in a rather straightforward manner obtained from De Bruijn's original asymmetrical version. We also will give a short proof of decreasing diagrams in the spirit of De Bruijn's proof, but with considerably less lemma's and diagrams to be checked.

Summarizing, De Bruijn's original version as the weak diamond property and Van Oostrom's decreasing diagram version are relatively close together, and can be seen as fairly straightforward consequences of each other. That the weak diamond property is a consequence of decreasing diagrams was already shown in [16]. The other direction is expounded in the present paper. It should be noted that the close proximity of the weak diamond property and decreasing diagrams does not detract from the value of Van Oostrom's key notion of decreasing diagrams, which employs a powerful invariant that is not at all visible yet in De Bruijn's version, nor his proof. This notion has proved to be very suitable for various extensions of the original decreasing diagram theorem [20, 19, 7].

We finally devote a closing section to the completeness issue for the present method by De Bruijn and the decreasing diagram version of Van Oostrom.

2. Weak Diamond Properties

In this section we introduce the various weak diamond properties as discussed by De Bruijn in his report. First, we will treat abstract reduction systems with just one reduction relation. In the next subsection, we enrich these structures with a labeling of steps. There we will state the weak diamond diagrams that are the subject of De Bruijn's memorandum, or rather, lead up to his main weak diamond theorem.

2.1. Unlabeled Reduction

Definition 2.1. An *abstract reduction system (ARS)* is a set A of *objects* together with a reduction relation $\to \subseteq A \times A$, called *one-step reduction*. Such an ARS is written as $(A, \to)$.

For general notions and notations about rewriting we refer to [15] or [10]. Here we mention a few of them: the transitive reflexive closure of $\to$ is written as $\twoheadrightarrow$ (sometimes also as $\to^*$, where * is Kleene's star, familiar from regular expressions). The reflexive closure of $\to$ is $\to^{\equiv}$. The converse of $\to$ is denoted by $\leftarrow$, and of $\twoheadrightarrow$ by $\twoheadleftarrow$. An object a for which there is no rewrite step $a \to \ldots$ possible is a *normal form* (but normal forms will not be an issue in this paper). Intuitively, $\to$ stands for the computation relation; a normal form is a 'final answer'.

Definition 2.2. A relation $\to \subseteq A \times A$ is *confluent* if $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$.

Confluence means that every pair of finite, co-initial rewrite sequences can be joined to a common reduct; see Figure 2. It is one of the most fundamental notions of (term) rewriting, guaranteeing the uniqueness of normal forms, if they exist.

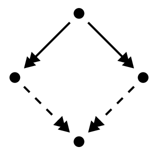


Figure 2: Confluence. In these diagrams, the solid arrows stand for universal (∀) quantification, while the dashed arrows stand for existential (∃) quantification.

Let us now follow De Bruijn's description in his introduction of memorandum 78-08. We combine the various rewrite properties in Figure 3, together with their defining diagram, their name as given by De Bruijn, and in some cases their modern names.

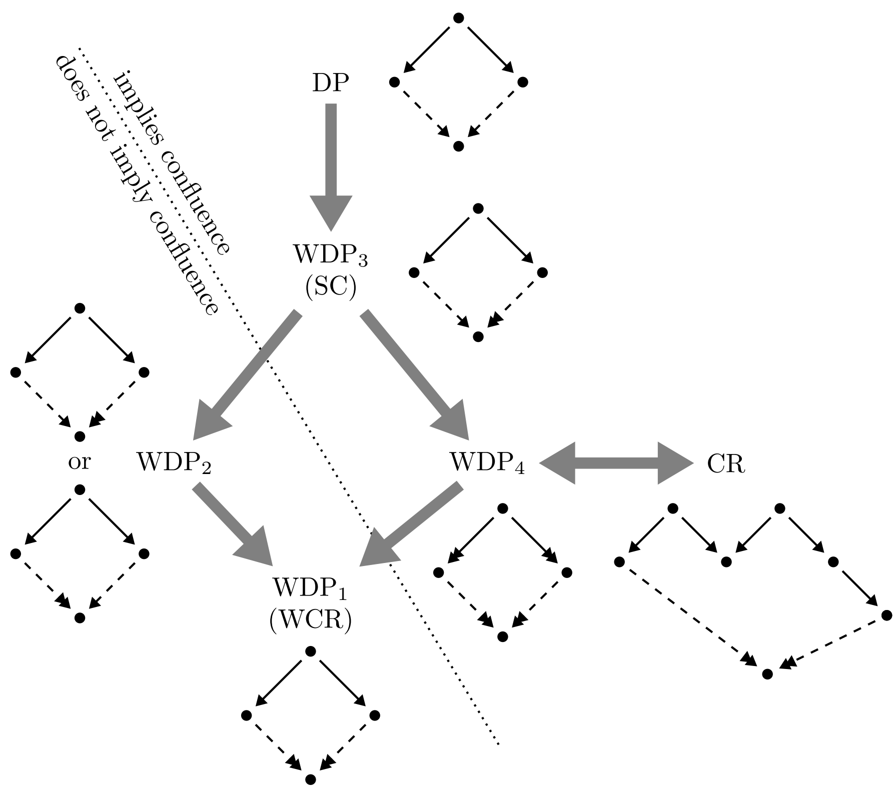


Figure 3: Various Confluence Properties.

Diagrams with single steps as initial diverging reductions are called *elemen-*

tary reduction diagrams. So except for WDP_4 and CR, all diagrams in Figure 3 are elementary diagrams. In the picture,

DP stands for *diamond property*,

WDP for *weak diamond property*,

CR for *Church-Rosser property*,

SC for *strong confluence* and

WCR for *weak Church-Rosser property* (or *local confluence*).

Actually, the defining diagrams in Figure 3 are just a convenient pictorial rendering of the more explicit definitions of the rewrite properties as mentioned. These explicit definitions can be most clearly stated in first-order predicate logic:

DP $\quad \forall a, b, c.\, \exists d.\, b \leftarrow a \rightarrow c \implies b \rightarrow d \leftarrow c$

$\mathrm{WDP}_1 \quad \forall a, b, c.\, \exists d.\, b \leftarrow a \rightarrow c \implies b \twoheadrightarrow d \twoheadleftarrow c$

$\mathrm{WDP}_2 \quad \forall a, b, c.\, \exists d.\, b \leftarrow a \rightarrow c \implies b \rightarrow d \twoheadleftarrow c \,\vee\, b \twoheadrightarrow d \leftarrow c$

$\mathrm{WDP}_3 \quad \forall a, b, c.\, \exists d.\, b \leftarrow a \rightarrow c \implies b \rightarrow d \twoheadleftarrow c$

$\mathrm{WDP}_4 \quad \forall a, b, c.\, \exists d.\, b \twoheadleftarrow a \twoheadrightarrow c \implies b \twoheadrightarrow d \twoheadleftarrow c$

CR $\quad \forall a, b.\, \exists c.\, a \leftrightarrow^* b \implies a \twoheadrightarrow c \twoheadleftarrow b$

Here $\leftrightarrow^*$ is the equivalence relation generated by $\rightarrow$, that is, its symmetric-transitive-reflexive closure, also called *convertibility.* In pictures as in Figure 3 convertibility is pictorially rendered as a *zigzag*, or *mountainscape*; involving some peaks and valleys. A *peak* consists of two diverging single steps. A convertibility is also called a *conversion.*

There is also a somewhat more succinct way of defining rewrite properties, namely by relational calculus. In a self-explaining notation, these definitions are as follows, involving the usual set-theoretic notions, together with '$\cdot$' referring to relational composition:

DP $\quad \leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$

$\mathrm{WDP}_1 \quad \leftarrow \cdot \rightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$

$\mathrm{WDP}_2 \quad \leftarrow \cdot \rightarrow \subseteq (\rightarrow \cdot \twoheadleftarrow) \cup (\twoheadrightarrow \cdot \leftarrow)$

$\mathrm{WDP}_3 \quad \leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \twoheadleftarrow$

$\mathrm{WDP}_4 \quad \twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$

CR $\quad \leftrightarrow^* \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$

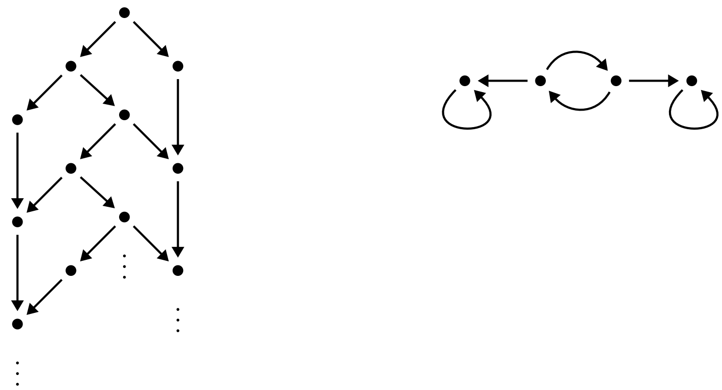


Figure 4: Two abstract reduction systems.

The dividing line in Figure 3 indicates which properties imply confluence and which do not. For the properties that do not imply confluence, this can be seen by inspection of counterexamples in Figure 4. The properties above the line in Figure 3 do not hold for the counterexamples. Actually the two ARSs in Figure 4 are in some sense 'equivalent', the one by De Bruijn to the left being the infinite unwinding of the finite one to the right.

Note the subtle difference between WDP_3 (SC) and WDP_2. When completing a confluence diagram, given the initial diverging reductions in order to arrive at converging reductions, in the former we can choose ourselves in which direction we have the splitting, see Figure 5. In the latter this choice is made for us. That WDP_3 (SC) implies CR, is easy to see; the reasoning was captured by De Bruijn by Figure 5 that we have copied from his memorandum.

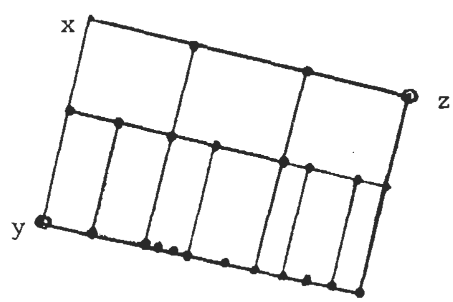


Figure 5: De Bruijn's illustration of diagram completion with WDP_3: splitting occurs only in one direction.

We are now ready to add more expressivity to the somewhat bare notion of an ARS $(A, \to)$ with just one reduction relation $\to$. To do this, we add *labels* or *indexes* $\alpha \in I$ to the relations, where I is some index set. De Bruijn himself suggests in the memorandum 78-08 that "indexed reduction relations may be used to represent reduction relations in the lambda calculus of various types, and diamond properties may depend on these types".

2.2. Labeled Reduction

After our introduction of the notions that do not yet involve labels, we now turn to the notions for labeled reductions. We start with defining an 'indexed' or 'labeled' abstract rewrite system.

Definition 2.3. An *indexed ARSs* $\mathcal{A} = (A, (\to_\alpha)_{\alpha \in I})$ consists of a set of A of *objects*, and a family $(\to_\alpha)_{\alpha \in I}$ of relations $\to_\alpha \subseteq A \times A$ indexed by some set I.

Let $(\to_\alpha)_{\alpha \in I}$ be a family of relations $\to_\alpha \subseteq A \times A$ indexed by I, and $> \subseteq I \times I$ an order on the index set. We define

$$\to \;=\; \bigcup_{\alpha \in I} \to_\alpha \qquad\qquad \to_{<\beta} \;=\; \bigcup_{\alpha < \beta} \to_\alpha \qquad\qquad \to_{\leq\beta} \;=\; \bigcup_{\alpha \leq \beta} \to_\alpha$$

for $\beta \in I$. Moreover, we write $\to_{<\alpha \text{ or } <\beta}$ as shorthand for $\to_{<\alpha} \cup \to_{<\beta}$.

We now arrive at the weak diamond property of De Bruijn which is based on the diagrams depicted in Figure 6. The colored arrows and lines inside the diagram are only meant as a visual aid in understanding: green lines stand for weak decrease (that is, $\geq$), red arrows for strict decrease (that is, $>$).

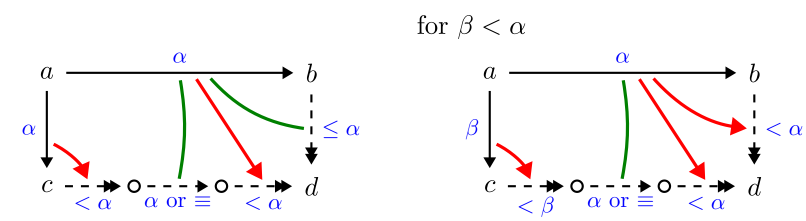


Figure 6: The weak diamond property. For peaks $c \leftarrow_\alpha a \to_\beta b$ with $\alpha = \beta$ the left diagram is required, and for $\beta < \alpha$ the right diagram..

Theorem 2.4 (Weak Diamond Property)**.** *Let $\mathcal{A} = (A, (\to_\alpha)_{\alpha \in I})$ be an ARS with reduction relations indexed by a well-founded* total *order $(I, >)$.*

Assume that for every peak $c \leftarrow_\beta a \to_\alpha b$ with $\alpha = \beta$ or $\beta < \alpha$, there exists an elementary diagram joining this peak of one of the forms shown in Figure 6. Then $\to$ is confluent.

In the present definition of De Bruijn, the weak diamond property requires a total order $>$, but this is not essential and in Section 4 we will see how it can be generalized to partial orders.

3. De Bruijn's Original Proof

We revive the truly amazing proof of Theorem 2.4 given by De Bruijn [5]. Large parts of the proof are rendered purely in the form of 'diagram pasting'. While each of the diagram completion steps is easy to check, the overall idea is far from obvious. The basis of the proof is formed by the diagrams shown in Figure 7; in De Bruijn's note they looked as in Figure 8. The diagrams $D_1(m)$ and $D_2(m, k)$ are those from the weak diamond property in Figure 6.

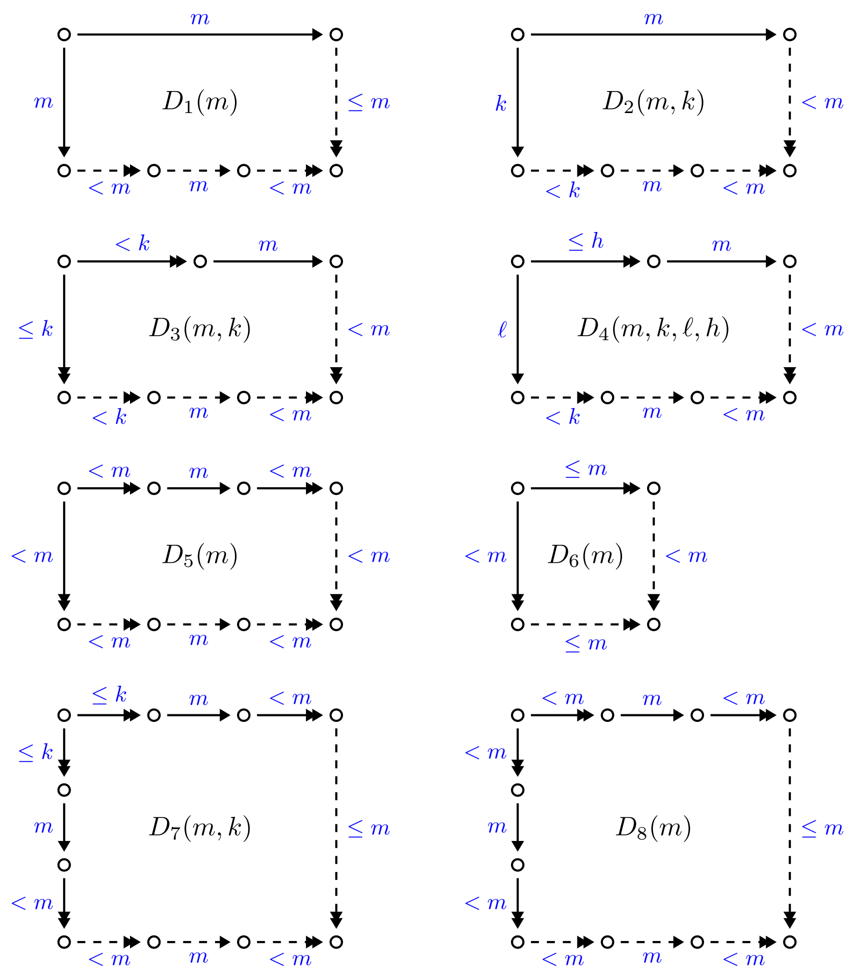


Figure 7: The diagrams in the proof by De Bruijn; here $h < k < m$ and $\ell \leq m$.

For large parts, we use the same wording and diagrams as De Bruijn. We have corrected a few typos concerning indices in Figure 11 and Lemma 3.6. We have slightly extended the proof Lemma 3.8 with an extra diagram in Figure 14 providing more details on the completion of the central part of the diagram.

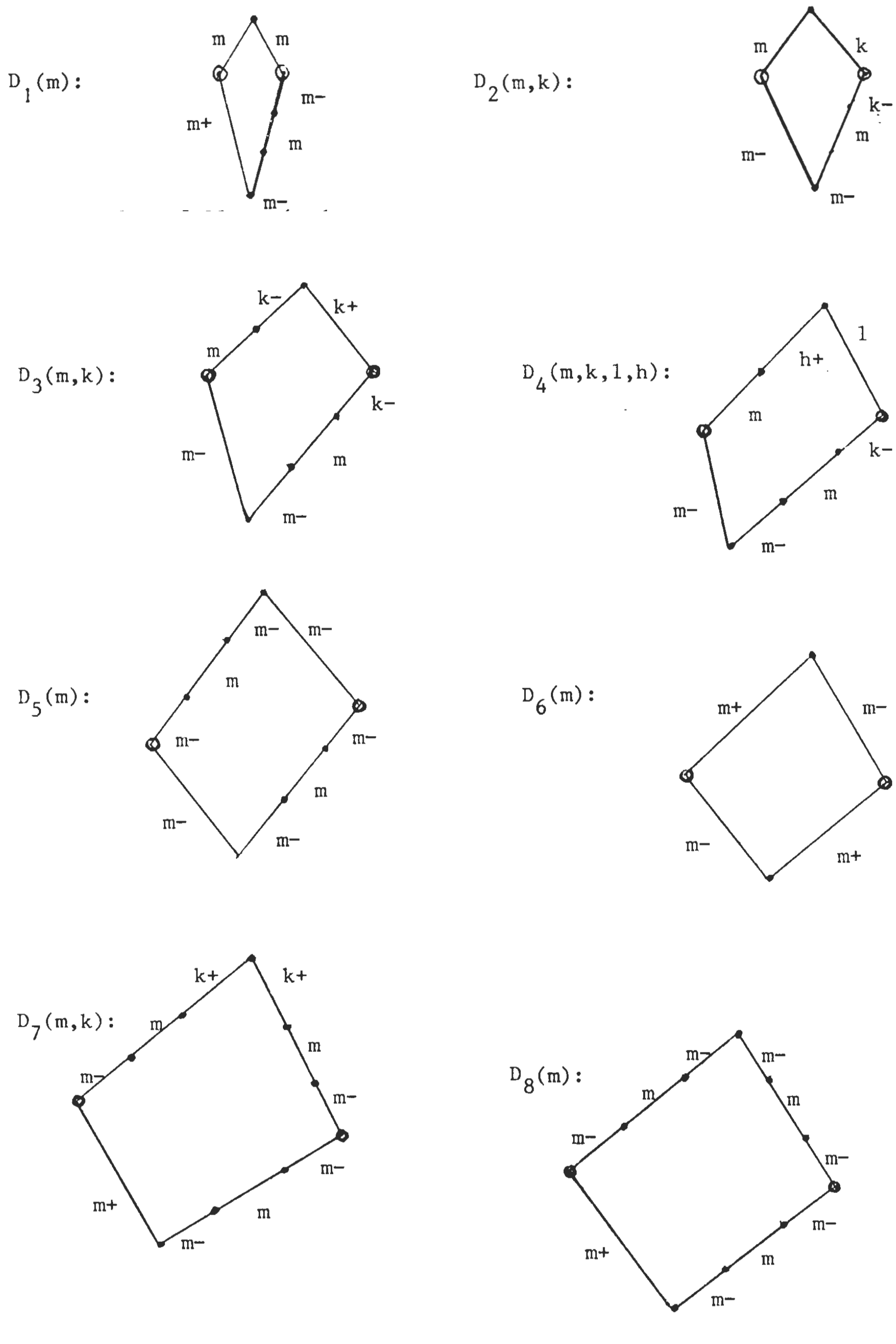


Figure 8: An excerpt from De Bruijn's original note.

The proof of De Bruijn [5] employs *Huet's strong confluence lemma* [8], see Figure 9, called WDP_3 or SC in Figure 3.

Lemma 3.1. *The property $\leftarrow \cdot \rightarrow \subseteq \twoheadrightarrow \cdot \leftarrow$ implies confluence of $\rightarrow$.*

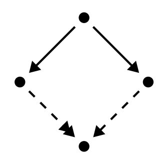


Figure 9: Strong Confluence.

Assumption. Throughout this section we assume that $D_1(m)$ holds for all m and $D_2(m,k)$ for all m, k with $k < m$.

Definition 3.2. For $m \in I$ we define

$$\begin{array}{rcl} \mathsf{CR}(m) & = & \twoheadleftarrow_{\leq m} \cdot \twoheadrightarrow_{\leq m} \subseteq \twoheadrightarrow_{\leq m} \cdot \twoheadleftarrow_{\leq m} \\ \mathsf{CR}^*(m) & = & \forall k < m.\, \mathsf{CR}(m) \end{array}$$

Then the following lemma is immediate.

Lemma 3.3. *If for all $m \in I$ we have $\mathsf{CR}(m)$, then we have CR.* □

In his note, De Bruijn erroneously states the equivalence of CR with $\forall m. \mathsf{CR}(m)$, but equivalence does not hold as is easily seen. His proof remains correct, as it only requires the implication $\Leftarrow$.

Lemma 3.4. *Let $m, k, \ell, h \in I$ with $h < k < m$ and $\ell \leq k$. Assume $D_3(m, \tau)$ for all $\tau \in I$ with $\tau < k$, and $D_3(\sigma, \tau)$ for all $\sigma, \tau \in I$ with $\tau < \sigma < m$. Furthermore assume $\mathsf{CR}^*(m)$. Then we have $D_4(m, k, \ell, h)$.*

Proof. For $\ell \leq h$, the argument is shown in Figure 10. For $h < \ell$ we use induction with respect to h by means of the diagram in Figure 11. Here IH stands for 'induction hypothesis'. □

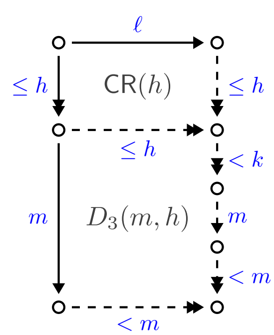


Figure 10: Proof of Lemma 3.4 for the case $\ell \leq h$.

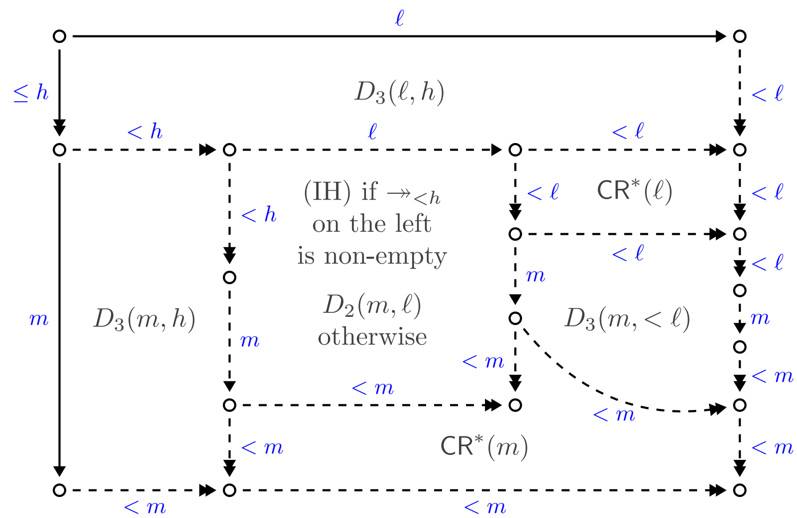


Figure 11: Proof of Lemma 3.4 for the case $\ell > h$.

Lemma 3.5. *Let $m, k \in I$ with $k < m$. Then we have*

$$(\ \mathsf{CR}^*(m) \ \ \wedge \ \ \forall \ell \leq k, h < k.\, D_4(m, k, \ell, h)\ ) \implies D_3(m, k)$$

Proof. The proof uses induction on the length of the reduction $\twoheadrightarrow_{\leq k}$ in the diagram $D_3(m, k)$. For length 0 there is nothing to be shown. For length $n + 1$ we have $\to_{\leq k}^{n+1} = \to_\ell \cdot \to_{\leq k}^{n}$ for some $\ell \leq k$. Then the peak of $D_3(m, k)$ can be completed as in Figure 12. $\square$

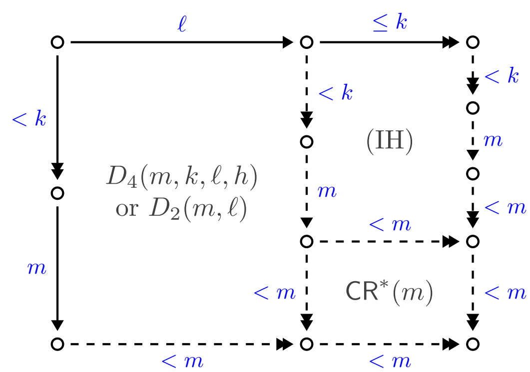


Figure 12: Proof of Lemma 3.5 where h is the maximum index of the rewrite sequence $\twoheadrightarrow_{<k}$ on the left; if empty, use $D_2(m, \ell)$.

Lemma 3.6. *Let $m \in I$. Assume $\mathsf{CR}^*(m)$ and assume that for all $h \in I$ with $h < m$, we have $D_3(m, h)$. Then $D_5(m)$.*

Proof. It suffices to prove the diagram in Figure 13 (left) for all $h < m$. (Note

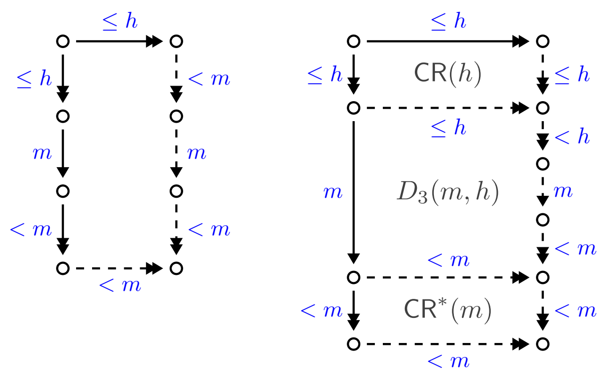


Figure 13: Proof of Lemma 3.6; diagram to be proven (left), and the proof (right).

that if m is the minimal element of I then the branches labeled with $< m$ or $\leq h$ are empty, and there is nothing to be proved.) The proof is given by Figure 13 (right). $\square$

Lemma 3.7. *For all $m \in I$, $D_5(m)$ implies $D_6(m)$.*

Proof. We have $\twoheadrightarrow_{\leq m} = (\twoheadrightarrow_{<m} \cdot \rightarrow_m \cdot \twoheadrightarrow_{<m})^*$. By induction on n and using $D_5(m)$ it follows that $(\twoheadleftarrow_{<m} \cdot \leftarrow_m \cdot \twoheadleftarrow_{<m})^n \cdot \twoheadrightarrow_{<m} \;\subseteq\; \twoheadrightarrow_{<m} \cdot \twoheadleftarrow_{\leq m}$ for all $n \in \mathbb{N}$. $\square$

Lemma 3.8. *Let $m, k \in I$ with $k < m$. Assume $\mathsf{CR}^*(m)$, $D_5(m)$, $D_6(m)$ and $D_3(m, h)$ for all $h < m$. Then we have $D_7(m, k)$.*

Proof. We apply induction with respect to k, assuming $D_7(m, h)$ for all $h < k$. Then the proof is displayed in Figure 14. $\square$

Lemma 3.9. *Let $m \in I$ and assume $D_7(m, k)$ for all $k < m$. Then we have $\mathsf{CR}(m)$.*

Proof. Since $D_7(m, k)$ for all $k < m$, we have the diagram $D_8(m)$ (if there are no $k < m$, we apply $D_1(m)$). Define $\rightsquigarrow \, = \, \twoheadrightarrow_{<m} \cdot \rightarrow_m \cdot \twoheadrightarrow_{<m}$ We have $\twoheadrightarrow_{\leq m} = \rightsquigarrow^*$, and by $D_8(m)$ we get $\leftsquigarrow \cdot \rightsquigarrow \; \subseteq \; \rightsquigarrow^* \cdot \leftsquigarrow$. Thus $\rightsquigarrow$ is confluent by Lemma 3.1, and hence we have $\mathsf{CR}(m)$. $\square$

Theorem 3.10. *For all $m \in I$ we have $\mathsf{CR}(m)$.*

Proof. Assume that the theorem is false. Then we have an m such that $\mathsf{CR}^*(m)$ is true, but $\mathsf{CR}(m)$ false. We cannot have $D_3(m, h)$ for all $h < m$, for then we would have $D_5(m)$ by Lemma 3.6, $D_6(m)$ by Lemma 3.7, and then Lemmas 3.8 and 3.9 would lead to $\mathsf{CR}(m)$. So there is some $k < m$ such that $D_3(m, k)$ is false. Let n be the smallest element of I for which $j \in I$ exists with $j < n$ and $D_3(n, j)$ false. Next let i be the smallest element such that $D_3(n, i)$ is false. Note that $n \leq m$ and $i < n$. By Lemma 3.5 we have ℓ and h such that $\ell \leq i$, $h < i$. and $D_4(n, i, \ell, h)$ is false. Now Lemma 3.4 leads to a contradiction. $\square$

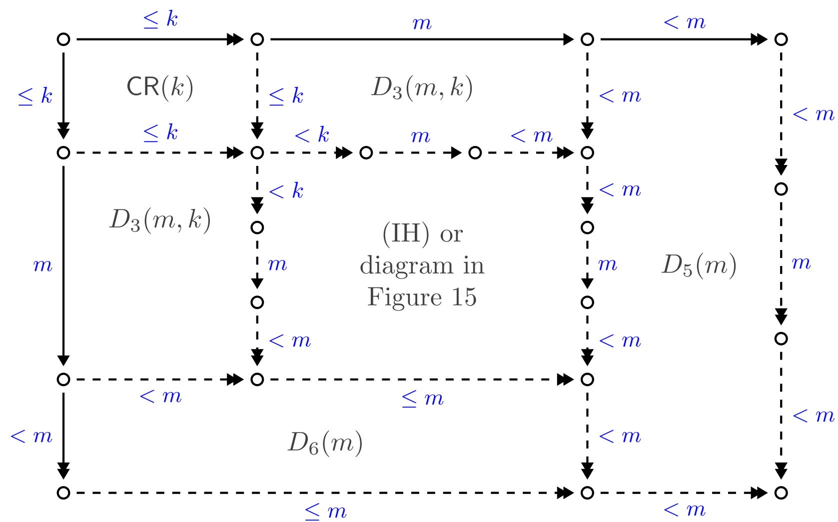


Figure 14: The proof of Lemma 3.8. The application of induction hypothesis (IH) takes h as the maximum label of the adjoining $\twoheadleftarrow_{<k} \cdot \twoheadrightarrow_{<k}$, and if this conversion is empty, we apply the diagram in Figure 15.

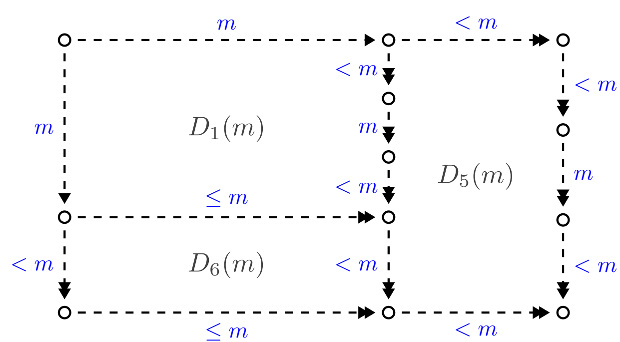


Figure 15: The proof of Lemma 3.8: the case of an empty peak $\twoheadleftarrow_{<k} \cdot \twoheadrightarrow_{<k}$.

4. Extending the Weak Diamond Property to Decreasing Diagrams

We will now show in two steps how De Bruijn's weak diamond property theorem (WDP) can be 'upgraded' to Van Oostrom's decreasing diagrams theorem (DD). The first step is relaxing the totality (or linearity) requirement on the order $>$. The second step symmetrizes the diagram so that we arrive at the proper form of an elementary decreasing diagram in Figure 20.

4.1. From Total to Partial Orders

As a first step, we perform a trivial adaptation of the weak diamond property which is vacuous for total orders, but yields a generalization after the switch to partial orders. As the diagram on the right in Figure 6 is only required for $\beta < \alpha$, we may replace '$< \alpha$' by '$< \alpha$ or $\leq \beta$'. The result is depicted in Figure 16 (adaptations are highlighted using a yellow marker).

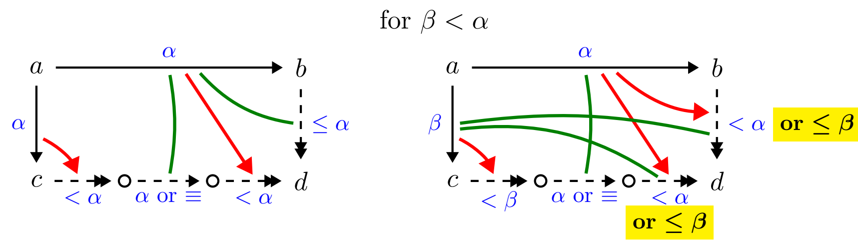


Figure 16: The weak diamond property with a, for total orders, vacuous adaptation.

In order to make the weak diamond property applicable for partial orders it suffices to replace the condition $\beta < \alpha$ by $\neg(\alpha \leq \beta)$. The result is shown in Figure 17. The idea is that any well-founded partial order $>$ can be extended to a well-founded total order $\succ$, and then $\beta \prec \alpha$ can hold only if not $\alpha \leq \beta$. We provide a more elaborate discussion of the extension in Remark 4.2 below.

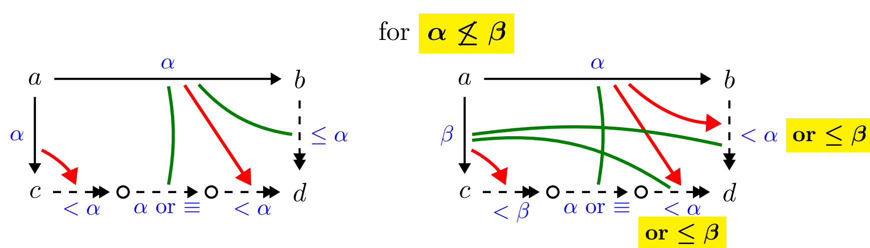


Figure 17: The weak diamond property for partial orders $(I, >)$; the changes with respect to the weak diamond property have been highlighted using a yellow marker.

We arrive at the following theorem:

Theorem 4.1 (Weak Diamond Property for Partial Orders)**.** *Let $\mathcal{A}$ be an indexed ARS $(A, (\rightarrow_\alpha)_{\alpha \in I})$ with reduction relations indexed by a well-founded* partial *order $(I, >)$. If for every peak of the form $c \leftarrow_\alpha a \rightarrow_\alpha b$ can be joined as in Figure 17 (left), and every peak $c \leftarrow_\beta a \rightarrow_\alpha b$ such that $\alpha \not\leq \beta$ and $\alpha \neq \beta$ can be joined as in Figure 17 (right), then $\rightarrow$ is confluent.* $\square$

Remark 4.2. While this upgrade to allow for partial orders is in essence elementary, at the same time there is some subtlety involved, and therefore we provide some more details of the reasoning. So we have a (well-founded) partial order $>$ on the labels $\alpha, \beta, \gamma, \ldots$, extended to a (well-founded) total or linear order $\succ$. We thus have the following facts:

(i) $\alpha < \beta \implies \alpha \prec \beta$
(ii) $\alpha = \beta \vee \beta \prec \alpha \vee \alpha \prec \beta$ (trichotomy for $\prec$)

(iii) $\alpha = \beta \ \vee \ \alpha \not\leq \beta \ \vee \ \beta \not\leq \alpha$ (trichotomy for $\leq$)

(iv) $\beta \prec \alpha \implies \alpha \not\leq \beta$

Here (i)–(iii) are immediate from the definitions, and (iv) follows simply: Suppose $\beta \prec \alpha$ and $\alpha \leq \beta$. Then $\beta \neq \alpha$ by irreflexivity of $\prec$, so $\alpha < \beta$, and hence $\alpha \prec \beta$, contradiction.

Now the key of the present argument is in the analogy between (ii) and (iii), together with (iv). Suppose that we have diagrams as in Figure 17 at our disposal to be used for pasting or tiling by inserting them in peaks. There are three types of tiles: the one with α versus α, the one with β versus α under the condition $\alpha \not\leq \beta$, and the symmetric one with β versus α under the condition $\beta \not\leq \alpha$. This corresponds to the trichotomy (iii) above, so we have enough tiles, all cases are covered. We are still in the framework with the partial order $<$, but now we extend $<$ to the total order $\prec$, for which we have the trichotomy (ii). Now we observe that in the process of extending $<$ to $\prec$, the requirements on the labels γ at the converging sides of the tiles, are translated neatly into the requirements for the tiles figuring in De Bruijn's weak diamond theorem. Let us check this. The left tile in Figure 17 is easy: $\gamma < \alpha$ becomes $\gamma \prec \alpha$, and $\gamma \leq \alpha$ becomes $\gamma \preceq \alpha$. The subtlety is in translating the rightmost tile in Figure 17 into a De Bruijn tile with the total order $\prec$. Consider a peak with labels α, β. If $\alpha = \beta$ we use the leftmost tile in Figure 17, which we recognized already to be a De Bruijn tile. Otherwise, by trichotomy for $\prec$, we have either $\beta \prec \alpha$ or $\alpha \prec \beta$. In the first case, we have by (iv): $\alpha \not\leq \beta$. Then we take accordingly the rightmost tile in Figure 17, which had the condition $\alpha \not\leq \beta$. Now we can 'transpose' the requirements: $\gamma < \beta$ becomes $\gamma \prec \beta$, and $\gamma < \alpha \ \vee \ \gamma \leq \beta$ becomes first $\gamma \prec \alpha \vee \ \gamma \ \ \preceq \beta$, and this literally boils down to just $\gamma \prec \alpha$ because $\beta \prec \alpha$ so that $\gamma \preceq \beta$ is superfluous ($\gamma \preceq \beta$ and $\beta \prec \alpha$ imply $\gamma \prec \alpha$). In the other case, $\alpha \prec \beta$, we perform symmetrically. The result is in both cases the rightmost De Bruijn tile in the framework of $\prec$, so that CR is implied.

Remark 4.3. In general, extending partial to total orders typically requires the Axiom of Choice (or equivalently Zorn's lemma). For the present situation, the extension of the weak diamond property to partial orders, we do not need this axiom or lemma. Namely, the order $(I, >)$ can be turned into a total order by replacing each $\alpha \in I$ by its order type (the ordinal corresponding to the height of α in the order $>$).

Remark 4.4. As a historical remark pertaining to the issue of upgrading from total to partial order, we mention that the second author included in his letter of March 5, 1990 to De Bruijn the diagrams as in Figure 18 where the order is now allowed to be partial. The assertion was stated that these diagrams also yield CR and indeed constitute a generalization of De Bruijn's weak diamond property. CR is implied if every peak can be closed by diagram 1, or diagram 2, or both diagrams 3 and 4, see below. For joining two converging reductions by pasting elementary diagrams this means: when confronted with a peak $\leftarrow_m \cdot \rightarrow_k$, we have either a 'tile' of the form 1 available, or one of the form 2, or we are given both a tile 3 and a tile 4 together. In that third case we have ourselves a choice to use tile 3 or 4. That De Bruijn's weak diamond property is a corollary of these diagrams is straightforward. We note that the present diagrams have an aspect also present in WDP_3 (SC), strong confluence; namely in the conjunction $3 \wedge 4$.

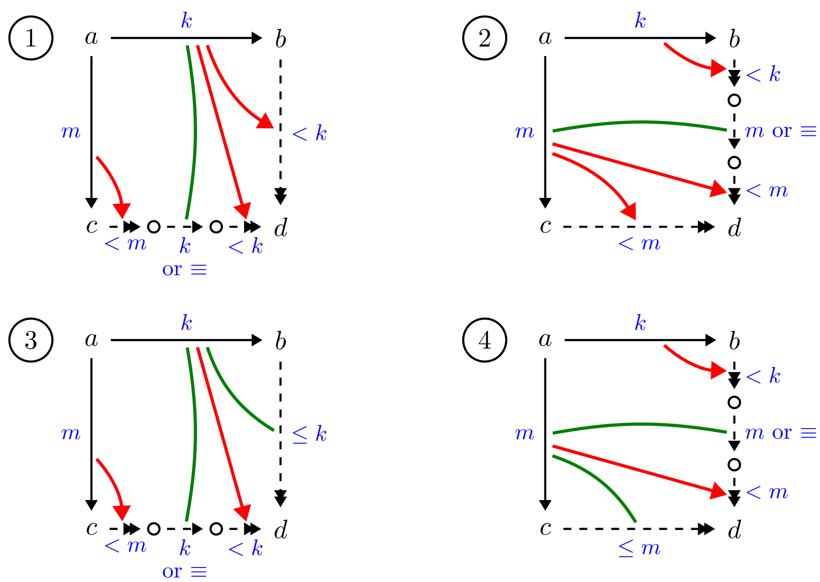
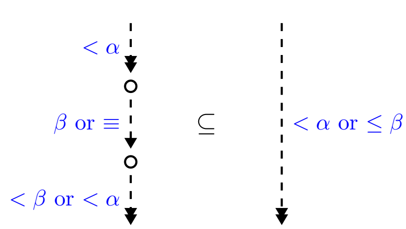


Figure 18: These diagrams entail CR as follows: $1 \vee 2 \vee (3 \wedge 4) \implies$ *CR. The diagrams 1 and 2 correspond with the right diagram in Figure 6 (single green line), while 3 and 4 correspond with the left diagram in Figure 6 (double green line, double red arrow).*

4.2. A Symmetric Diagram

We perform two transformation steps in order to obtain a symmetric version. First, we fuse the two diagrams in Figure 17 to a single diagram. Allowing $\alpha = \beta$ in the diagram on the right, we obtain the diagram on the left, except that $\leq \beta$ in the bottom reduction must be replaced by the stricter condition $< \beta$. In doing so, we obtain the diagram shown in Figure 19.

Finally, we obtain a symmetric diagram by noting that



for $\alpha \not< \beta$

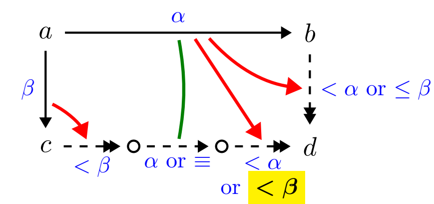


Figure 19: A single diagram version of the weak diamond property for partial orders.

which allows us to change the right-hand side of the diagram in Figure 19. Due to the symmetry we can also drop the condition $\alpha \not< \beta$. The resulting diagram in Figure 20 is known as Van Oostrom's *decreasing elementary diagram* introduced in Van Oostrom [16, 17].

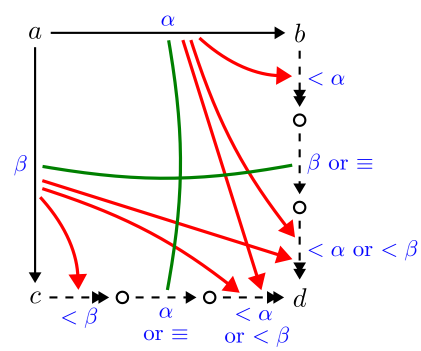


Figure 20: Decreasing elementary diagram.

Theorem 4.5 (Decreasing Diagrams)**.** *Let $\mathcal{A} = (A, (\to_\alpha)_{\alpha \in I})$ be an ARS with reduction relations indexed by a well-founded* partial *order $(I, >)$.*

Assume that for every peak $c \leftarrow_\beta a \to_\alpha b$ there exists an elementary diagram joining this peak of the form shown in Figure 20. Then $\to$ is confluent.

Remark 4.6. Both the weak diamond property and decreasing diagrams treated in this paper rely on a labeling of the *edges* in a reduction graph (rewrite steps). Mirna Bognar remarked that it is also possible to label the *points* $a \in A$ of the ARS $(A, \to)$ rather than the steps $a \to a'$. Such a *point-labeled version* of the decreasing diagram theorem was given by her in the technical report [4].

4.3. Proving Decreasing Diagrams using the Weak Diamond Property

In Sections 4.1 and 4.2 we have seen how to derive the elementary decreasing diagram from the weak diamond property. If we are only interested in a proof, then the following shorter argument suffices.

Roughly speaking, when switching from a partial to a total order, the structure on one of the sides of the decreasing diagram disappears, and the decreasing diagram becomes a witness of the weak diamond property.

Proof of Theorem 4.5. Let $(A, (\to_\alpha)_{\alpha\in I})$ and $(I, >)$ fulfill the conditions of Theorem 4.5. Let $\succ \subseteq I \times I$ be a total well-founded such that $> \subseteq \succ$. Consider a peak $c \leftarrow_\beta a \to_\alpha b$ together with a decreasing diagram joining this peak. For $\beta = \alpha$ we obtain

$$\begin{array}{lll} \text{lower side:} & \twoheadrightarrow_{<\beta} \circ \to_\alpha^{\equiv} \circ \twoheadrightarrow_{<\alpha \text{ or } <\beta} & \subseteq \; \twoheadrightarrow_{\prec\alpha} \circ \to_\alpha^{\equiv} \circ \twoheadrightarrow_{\prec\alpha} \\ \text{right side:} & \twoheadrightarrow_{<\alpha} \circ \to_\beta^{\equiv} \circ \twoheadrightarrow_{<\alpha \text{ or } <\beta} & \subseteq \; \twoheadrightarrow_{\preceq\alpha} \end{array}$$

If $\beta \neq \alpha$, we have $\beta \prec \alpha$ or $\beta \succ \alpha$. By symmetry we assume that $\beta \prec \alpha$. Then

$$\begin{array}{lll} \text{lower side:} & \twoheadrightarrow_{<\beta} \circ \to_\alpha^{\equiv} \circ \twoheadrightarrow_{<\alpha \text{ or } <\beta} & \subseteq \; \twoheadrightarrow_{\prec\beta} \circ \to_\alpha^{\equiv} \circ \twoheadrightarrow_{\prec\alpha} \\ \text{right side:} & \twoheadrightarrow_{<\alpha} \circ \to_\beta^{\equiv} \circ \twoheadrightarrow_{<\alpha \text{ or } <\beta} & \subseteq \; \twoheadrightarrow_{\prec\alpha} \end{array}$$

Thus, in each case, the elementary decreasing diagram for $c \leftarrow_\beta a \to_\alpha b$ with respect to $>$ is a witness of the weak diamond property with respect to $\succ$. $\square$

Remark 4.7. Let us mention a related remark of Hans Zantema (personal communication during the PhD promotion of Van Oostrom in 1994). Hans Zantema observed that a(n elementary) decreasing diagram stays valid when the order $>$ on the labels is extended. Thus the decreasing diagrams theorem only becomes stronger when extending the well-founded order. Since every well-founded order can be extended to a well-founded total order (using the Axiom of Choice), it suffices to prove the decreasing diagrams theorem for well-founded total orders. Hans Zantema posed the question whether totality could be exploited to simplify the proof?

In some sense, we give a positive answer to this question. We observe that when switching to a total order, the elementary decreasing diagrams become the diagrams of the weak diamond property. Thus the decreasing diagrams theorem can be obtained as a direct consequence of the weak diamond property (see the proof above).

Again, as in Remark 4.3, for embedding the partial order into a total order, we can do without the Axiom of Choice (or the lemma of Zorn).

5. A Proof of Decreasing Diagrams à la De Bruijn

In this section, we give a short proof that decreasing diagrams implies confluence[1]. Similar to De Bruijn's original proof, we employ a nested induction and diagram pasting. However, our proof uses significantly less lemmas and diagrams.

[1]We note that the proof can easily be adapted for the weak diamond property, yielding an alternative proof for the weak diamond property, but still in the spirit of De Bruijn's method. We leave the adaptation to the reader.

Proof of Confluence by Decreasing Diagrams. Let $\mathcal{A} = (A, (\to_\alpha)_{\alpha \in I})$ with a well-founded order $(I, >)$, and assume that $(*)$ for every peak $c \leftarrow_\beta a \to_\alpha b$ we have a decreasing diagram. Without loss of generality we may assume that $>$ is total, $\mathsf{id}_A \subseteq \to_\alpha$ for all $\alpha \in I$, and we have a bottom element $\bot \in I$ and a top element $\top \in I$ with $\to_\bot = \to_\top = \mathsf{id}_A$. The assumptions help to reduce case distinctions in the proof. For $\alpha \in I$, we define $\rightsquigarrow_\alpha = \twoheadrightarrow_{<\alpha} \cdot \to_\alpha \cdot \twoheadrightarrow_{<\alpha}$.

For $\alpha, \beta \in I$, we let $X(\alpha, \beta)$ be the following diagram:

$$\begin{tikzcd}
\circ \arrow[rr, "\alpha"] \arrow[d, "\leq \beta"'] \arrow[drrr, phantom, "X(\alpha,\beta)"] & & \circ \arrow[r, two heads, "<\alpha"] & \circ \arrow[d, dashed, two heads, "<\alpha \text{ or } \leq \beta"] \\
\circ \arrow[r, dashed, two heads, "<\beta"'] & \circ \arrow[r, dashed, "\alpha"'] & \circ \arrow[r, dashed, two heads, "<\alpha"'] & \circ
\end{tikzcd}$$

For every $\alpha \in I$ we have that $X(\alpha, \alpha)$ implies $\twoheadleftarrow_{\leq\alpha} \cdot \to_{\leq\alpha} \subseteq \twoheadrightarrow_{\leq\alpha} \cdot \twoheadleftarrow_{\leq\alpha}$, and thus confluence of $\to_{\leq\alpha}$. Hence, for confluence of $\to$ it suffices to prove $X(\top, \top)$.

We show $X(\alpha, \beta)$ for all α, β with $\beta \leq \alpha$ by well-founded induction on (α, β) in the lexicographic order with respect to $>$. Thus we may assume $X(\alpha', \beta')$ for all $\beta' \leq \alpha' \leq \alpha$ such that either $\alpha' < \alpha$ or $\beta' < \beta$. Then we have:

($\S_1$) The relation $\to_{\leq\gamma}$ is confluent for all $\gamma < \alpha$.

We prove the following auxiliary claim:

($\S_2$) $\twoheadleftarrow_{<\beta} \cdot \rightsquigarrow_\beta^n \subseteq \rightsquigarrow_\beta^n \cdot \twoheadleftarrow_{<\beta}$ for all $n \in \mathbb{N}$.

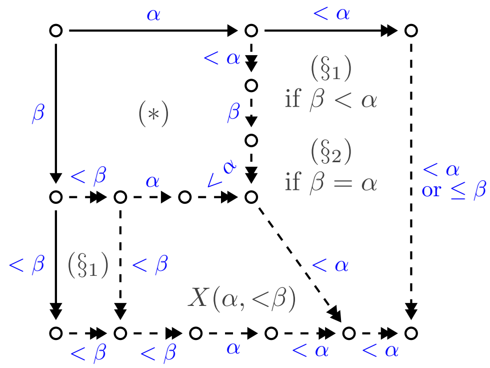
By induction on n, we only need to consider the case $n = 1$. The proof is given by the following diagram:

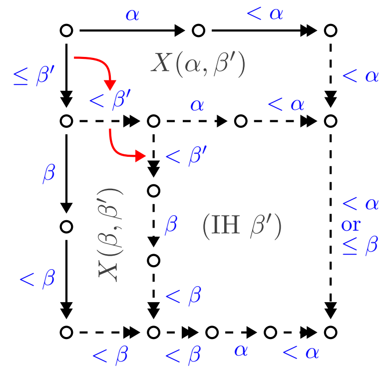
$$\begin{tikzcd}
\circ \arrow[r, two heads, "<\beta"] \arrow[d, "<\beta"'] \arrow[dr, phantom, "(\S_1)"] & \circ \arrow[rr, "\beta"] \arrow[d, dashed, two heads, "<\beta"] \arrow[drrr, phantom, "X(\beta,<\beta)"] & & \circ \arrow[r, two heads, "<\beta"] & \circ \arrow[d, dashed, two heads, "<\beta"] \\
\circ \arrow[r, dashed, two heads, "<\beta"'] & \circ \arrow[r, dashed, two heads, "<\beta"'] & \circ \arrow[r, dashed, "\beta"'] & \circ \arrow[r, dashed, two heads, "<\beta"'] & \circ
\end{tikzcd}$$

This proves claim ($\S_2$).

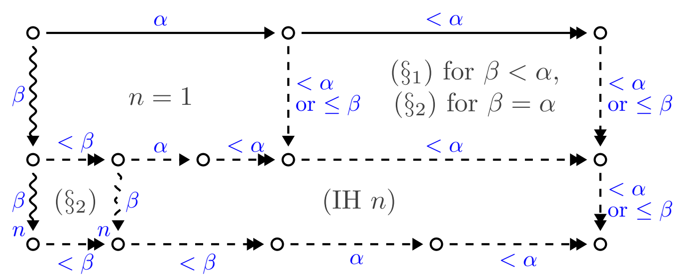
Note that $\twoheadrightarrow_{\leq\beta} = \rightsquigarrow_\beta^*$. To prove $X(\alpha, \beta)$, we use induction on the length n of the rewrite sequence $\sigma : \rightsquigarrow_\beta^n$ on the left of $X(\alpha, \beta)$.

- For the base case $n = 1$, we have that σ is of the form $\sigma : \twoheadrightarrow_{\leq\beta'} \cdot \to_\beta \cdot \twoheadrightarrow_{<\beta}$ with $\beta' < \beta$. We use induction on β'. For an empty reduction $\twoheadrightarrow_{\leq\beta'}$ (and equivalently $\beta' = \bot$) the proof is given on the left, for $\beta' > \bot$ on the right:





- The induction step $n \mapsto n+1$ is as follows:



This concludes the proof of confluence by decreasing diagrams.

6. Completeness

6.1. Completeness for Confluence

The weak diamond property and the decreasing diagram theorem have many applications. Newman's Lemma [12], stating that WCR $\wedge$ SN $\implies$ CR where SN (strong normalization) is the property that there are no infinite reductions sequences $a_0 \to a_1 \to a_2 \ldots$ in $(A, \to)$ is a corollary, and so is Huet's SC (strong confluence) lemma. Several other well-known abstract confluence lemma's are also corollaries [16, 15]. Thus the question arises how far this method to prove CR reaches. To consider this question we define:

Definition 6.1. Let $\mathcal{A} = (A, \to)$ be an unlabeled ARS, and $\mathcal{A}' = (A, (\to_\alpha)_{\alpha \in I})$ be a labeled ARS with the same objects and labeled reduction relations $\to_\alpha$ such that $\to = \bigcup_{\alpha \in I} \to_\alpha$. Then $\mathcal{A}'$ is called *labeled version of* $\mathcal{A}$.

We state the following definition for decreasing diagrams, but it generalizes in the obvious way to the weak diamond property.

Definition 6.2. An ARS $\mathcal{A}$ has the property DCR, *decreasing Church-Rosser*, if there is a labeled version $\mathcal{A}'$ of $\mathcal{A}$ for which we have decreasing elementary diagrams (that is, every peak $c \leftarrow_\beta a \to_\alpha b$ can be closed in the form of a decreasing elementary diagram).

So the decreasing diagrams technique can now be stated as

$$\text{DCR} \implies \text{CR}$$

And this raises the question whether also the reverse CR $\implies$ DCR holds, in other words, whether the decreasing diagram method, or the weak diamond property, is *complete*: DCR $\iff$ CR.

Van Oostrom [17] has shown that this equivalence holds for *countable* $\mathcal{A}$. For general $\mathcal{A}$ this question is still open. The completeness for countable ARSs $\mathcal{A}$ is a corollary of the decreasing diagram theorem together with the theorem in [10] stating that for countable ARSs $\mathcal{A}$ we have CR $\iff$ CP where CP (cofinality property) asserts for all $a \in A$, the existence of a *cofinal* reduction sequence $a = a_0 \to a_1 \to a_2 \ldots$ that overtakes every finite reduction of a in the following sense: if $a \twoheadrightarrow a'$, then there exists $n \in \mathbb{N}$ such that $a' \twoheadrightarrow a_n$.

It is easy to see that CP $\implies$ CR always holds. And if $\mathcal{A}$ is countable, it is not hard to use CR to construct for every $a \in A$ a cofinal reduction. For uncountable $\mathcal{A}$ this is not always possible: [9] gives the counterexample $\mathcal{A} = (\aleph_1, \to)$ where $\aleph_1$ is the first uncountable cardinal number, which is by definition the set of countable ordinals, and where the reduction relation $\to$ is defined by $\alpha \to \beta \iff \alpha < \beta$. This $\mathcal{A}$ is CR, but CP fails, due to elementary properties of ordinals ($\aleph_1$ does not have cofinality character ω). Note that $(\aleph_1, <)$ does satisfy DCR, since we have DP (diamond property).

In fact we have for every ARS CP $\implies$ DCR. For the proof we refer to [15, Prop. 14.2.30]. So, summarizing the above, we have for countable ARSs:

$$\text{CP} \iff \text{DCR} \iff \text{CR}$$

The completeness of DCR for countable ARSs is a pleasant state of affairs, since most 'relevant' ARSs will be countable, among them ARSs obtained from term rewriting systems over a countable signature, such as various lambda calculi, the original concern of De Bruijn.

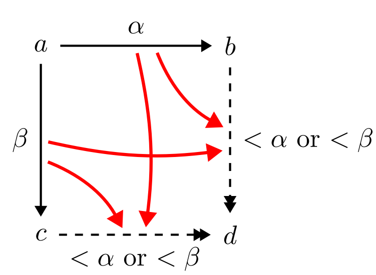


Figure 21: Strictly decreasing diagram: a simple, complete elementary diagram.

Actually, when we are looking for a method to prove CR that is complete for countable ARSs, we can adopt a slightly simpler form of decreasing diagrams, that we call a *strictly decreasing diagram* depicted in Figure 21. We write SDCR (*strictly decreasing Church-Rosser*) if we have DCR on the basis of strictly decreasing diagrams (which also are decreasing diagrams, trivially). An inspection of the proof of [15, Prop. 14.2.30], to which we just referred, shows that we can upgrade our completeness statement above as follows: For all countable ARSs,

$$\mathrm{CP} \iff \mathrm{SDCR} \iff \mathrm{DCR} \iff \mathrm{CR}$$

As mentioned above, the following remains an open problem:

Open Problem 6.3. Is the decreasing diagram method complete:

$$\mathrm{DCR} \iff \mathrm{CR}\,?$$

Following Van Oostrom, we conjecture that the answer is *no*. As a step towards a counterexample for completeness it may be fruitful to first show that the simpler strictly decreasing diagrams technique is not complete in general:

Question 6.4. Is the strictly decreasing diagram method complete:

$$\mathrm{SDCR} \iff \mathrm{CR}\,?$$

6.2. Incompleteness for Commutation

In this subsection we will mention an observation about commuting reduction relations. We define:

Definition 6.5. Let $\to, \rightsquigarrow \subseteq A \times A$ be relations on a set A. Then $\to$ *commutes with* $\rightsquigarrow$ if $\leftarrow^* \cdot \rightsquigarrow^* \subseteq \rightsquigarrow^* \cdot \leftarrow^*$.

Again following Van Oostrom, given a situation with reduction relations $\to, \rightsquigarrow$, it is natural to attempt to prove commutation in a similar fashion as in this paper, by elementary commutation diagrams involving the relations $\to, \rightsquigarrow$, and pasting of such diagrams. *Elementary diagrams for commutation* consist of peaks of the form $\leftarrow \cdot \rightsquigarrow$ with joining reductions $\rightsquigarrow^* \cdot \twoheadleftarrow$.

The decreasing diagram technique can also be used for proving commutation, see [16]; the elementary decreasing diagram for commutation looks as shown in Figure 22.

As we have seen, the decreasing diagrams technique is complete for countable systems. For the more general commutation, the situation is more involved. We will see that any method that tries to complete diverging reductions $\leftarrow^* \cdot \rightsquigarrow^*$ using only local commutation diagrams is bound to be incomplete, even for finite abstract rewrite systems. In particular, no such method can complete the diverging reductions $a' \leftarrow a \rightsquigarrow b \rightsquigarrow b'$ in the rewrite system shown in Figure 23.

An infinite counterexample was found at the International School on Rewriting 2008 (ISR) in Obergurgl, Austria by the first author and Clemens Grabmayer; it was subsequently simplified by Vincent van Oostrom to the finite one shown in Figure 23. Note the resemblance for the case of single reduction relations (not labeled) with the ARS in Figure 4.

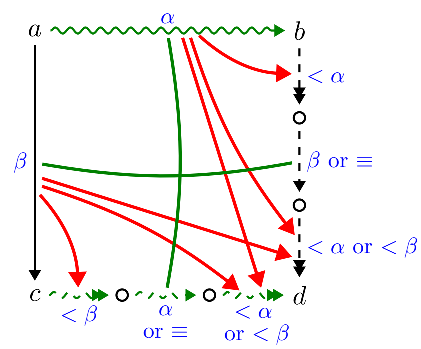


Figure 22: Decreasing elementary diagram for proving commutation.

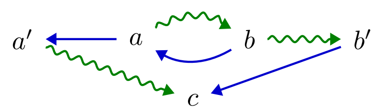


Figure 23: Incompleteness for commutation.

Theorem 6.6. *Any method for proving commutation, that is based on joining coinitial reductions by pasting elementary commutation diagrams, is incomplete (even for finite abstract reduction systems).*

Proof. We consider the ARS in Figure 23, that is $a \rightsquigarrow b \rightsquigarrow b' \to c$ and $b \to a \to a' \rightsquigarrow c$. It is easy to check that $\to$ commutes with $\to$. Nevertheless, the coinitial reductions $a' \leftarrow a \rightsquigarrow b \rightsquigarrow b'$ cannot be joined by pasting local commutation diagrams.

$$\begin{tikzcd} a \arrow[r, rightsquigarrow] \arrow[d] & b \arrow[r, rightsquigarrow] & b' \\ a' & & \end{tikzcd}$$

The only way to join the peak $a' \leftarrow a \rightsquigarrow b$ using reductions of the form $\rightsquigarrow^* \cdot \twoheadleftarrow$ is $a' \leftarrow a \leftarrow b$. Thus we obtain $a' \leftarrow a \leftarrow b \rightsquigarrow b'$:

$$\begin{tikzcd} a \arrow[r, rightsquigarrow] \arrow[dd] & b \arrow[r, rightsquigarrow] \arrow[d] & b' \\ & a \arrow[d] & \\ a' \arrow[r, dash, "\equiv"'] & a' & \end{tikzcd}$$

Now the peak $a \leftarrow b \rightsquigarrow b'$ can only be joined by $a \rightsquigarrow b \rightsquigarrow b'$:

$$
\begin{tikzcd}
a \arrow[r, rightsquigarrow] \arrow[dd] & b \arrow[rr, rightsquigarrow] \arrow[d] & & b' \arrow[d, dash, "\equiv"] \\
& a \arrow[r, rightsquigarrow] \arrow[d] & b \arrow[r, rightsquigarrow] & b' \\
a' \arrow[r, dash, "\equiv"'] & a' & &
\end{tikzcd}
$$

Now we are exactly where we started from: $a' \leftarrow a \rightsquigarrow b \rightsquigarrow b'$. Thus there is no way to join this reduction using local commutation diagrams. □

Acknowledgments. The treatment in our paper has profited greatly over an extended range of years from useful discussions with Hans Zantema, Roel de Vrijer, Aart Middeldorp and Vincent van Oostrom, for which we express our gratitude.

References

[1] H.P. Barendregt. *The Lambda Calculus. Its Syntax and Semantics*, volume 103 of *Studies in Logic and The Foundations of Mathematics*. North-Holland, revised edition, 1984.

[2] H.P. Barendregt, W. Dekkers, and R. Statman. *Lambda Calculus with Types*. Perspectives in Logic. Cambridge University Press, 2013.

[3] M. Bezem, J.W. Klop, and V. van Oostrom. Diagram Techniques for Confluence. *Information and Computation*, 141(2):172–204, 1998.

[4] M. Bognar and J.W. Klop. A Note on some Abstract Confluence Criteria. Technical Report IR-411, Vrije Universiteit Amsterdam, 1996.

[5] N.G. de Bruijn. A Note on Weak Diamond Properties. Memorandum 78–08, Eindhoven Uninversity of Technology, 1978.

[6] B. Felgenhauer and V. van Oostrom. Proof Orders for Decreasing Diagrams. In *Proc. Conf. on Rewriting Techniques and Applications (RTA 2013)*, volume 21 of *LIPIcs*, pages 174–189. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013.

[7] N. Hirokawa and A. Middeldorp. Decreasing Diagrams and Relative Termination. *J. Autom. Reasoning*, 47(4):481–501, 2011.

[8] G.P. Huet. Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems. *Journal of the ACM*, 27(4):797–821, 1980.

[9] J. W. Klop. *Combinatory Reduction Systems*, volume 127 of *Mathematical Centre Tracts*. Mathematisch Centrum, Amsterdam, 1980.

[10] J. W. Klop. Term rewriting systems. In S. Abramsky, D. Gabbay, and T. Maibaum, editors, *Handbook of Logic in Computer Science*, volume II, pages 1–116. Oxford University Press, 1992.

[11] J.W. Klop, V. van Oostrom, and R.C de Vrijer. A Geometric Proof of Confluence by Decreasing Diagrams. *J. Log. Comput.*, 10(3):437–460, 2000.

[12] M.H.A. Newman. On Theories with a Combinatorial Definition of "Equivalence". *Annals of Mathematics*, 42(2):223–243, 1942.

[13] J.H. Geuvers R.P. Nederpelt and R.C. de Vrijer, editors. *Selected Papers on Automath*, volume 133 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 1994.

[14] M.H. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard Isomorphism*. Number v. 10 in Studies in Logic and the Foundations of Mathematics. Elsevier, 2006.

[15] Terese. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003.

[16] V. van Oostrom. Confluence by Decreasing Diagrams. *Theoretical Computer Science*, 126(2):259–280, 1994.

[17] V. van Oostrom. *Confluence for Abstract and Higher-Order Rewriting*. PhD thesis, Vrije Universiteit Amsterdam, 1994.

[18] V. van Oostrom. Confluence by Decreasing Diagrams, Converted. In *Proc. Conf. on Rewriting Techniques and Applications (RTA 2008)*, volume 5117 of *LNCS*, pages 306–320. Springer, 2008.

[19] H. Zankl, B. Felgenhauer, and A. Middeldorp. CSI - A Confluence Tool. In *Proc. Conf. on Automated Deduction (CADE 2011)*, volume 6803 of *LNCS*, pages 499–505. Springer, 2011.

[20] H. Zankl, B. Felgenhauer, and A. Middeldorp. Labelings for Decreasing Diagrams. In *Proc. Conf. on Rewriting Techniques and Applications (RTA 2011)*, volume 10 of *LIPIcs*, pages 377–392. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2011.