Confluence of the Chinese Monoid

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Abstract. The Chinese monoid, related to Knuth's Plactic monoid, is of great interest in algebraic combinatorics. Both are ternary monoids, generated by relations between words of three symbols. The relations are, for a totally ordered alphabet, $cba = cab = bca$ if $a \leq b \leq c$. In this note we establish confluence by tiling for the Chinese monoid, with the consequence that every two words u, v have extensions to a common word: $\forall u, v.\ \exists x, y.\ ux = vy$.
Our proof is given using decreasing diagrams, a method for obtaining confluence that is central in abstract rewriting theory. Decreasing diagrams may also be applicable to various related monoid presentations. We conclude with some open questions for the monoids considered.

Dedication. *Our paper is dedicated in friendship to Catuscia Palamidessi for her 60th anniversary, with fond memories of the second author of cooperations during her stays around 1990 at CWI Amsterdam; with admiration for her work and accomplishments.*

1 Introduction

This paper is concerned with the Chinese monoid which is the quotient of the free monoid over a totally ordered alphabet with respect to the congruence generated by

$$cba = cab = bca \quad \text{for every } a \leq b \leq c\,.$$

These relations are equivalent to

$$\begin{aligned} aba = baa,\ bba = bab &\quad \text{for every } a < b, \\ cab = cba = bca &\quad \text{for every } a < b < c\,. \end{aligned} \tag{1}$$

The Chinese monoid plays an important role in algebra and combinatorics. It is closely related to the Plactic monoid of Knuth [16] generated by

$$\begin{aligned} aba = baa,\ bba = bab &\quad \text{for every } a < b, \\ cab = acb,\ bca = bac &\quad \text{for every } a < b < c\,. \end{aligned} \tag{2}$$

For the case of two generators, the Chinese monoid coincides with the Plactic monoid. Knuth devised the equations (2) in 1970 to analyse Schensted's algorithm [21] for finding the longest increasing subsequence of a sequence of integers.

The term ‘Plactic monoid’ has been coined by Lascoux & Schützenberger [18, 17]. Their theory of the Plactic monoid became an important tool in various combinatorial contexts (see further [4, 3, 12]). For an analysis of monoids using advanced rewriting techniques, see [11, 19]. A quantum perspective on the Plactic monoid has been discovered by Date, Jimbo and Miwa [5], 20 years after the equations have been suggested by Knuth. In 1995, Leclerc and Thibon [20] have obtained a quantum characterisation of the Plactic monoid, showing that the Plactic monoid can be interpreted as a maximal torus for the quantum group $U_q(gl(n, C))$. It is likely that the Chinese monoid plays a similar role for another quantum group (see further [5]).

Our contribution

In this note we use decreasing diagrams, a technique from abstract rewriting theory, to establish confluence by tiling for the Chinese monoid. So for all words u, v there exist extensions x, y such that $ux = vy$.

Abstract rewriting theory is an initial part of term rewriting (see [8]) where the structure of the objects is disregarded. An abstract reduction system is just a set equipped with binary ‘reduction’ relations. A seminal result in abstract rewriting is the classical Newman’s lemma that yields confluence (CR) as a consequence of termination (SN) together with local confluence (WCR or weak Church-Rosser).

Decreasing diagrams ([6, 23, 22, 8, 10]) is a method to prove CR that vastly improves Newman’s Lemma, which is one its many corollaries. The technique employs a labelling of the steps with labels from a well-founded partial order in order to conclude confluence of the underlying unlabelled relation. Decreasing diagrams are complete for proving confluence of countable systems.[3] The challenge typically is finding a suitable (decreasing) labelling of the steps.

Somewhat surprisingly, for the Chinese monoid the natural labelling of the steps turns out to be immediately suitable for the application of decreasing diagrams. This is in contrast to the Plactic monoid and the Braid monoid. For $n > 2$ the usual monoid presentation for the Plactic monoid does not admit a straightforward confluence by tiling proof via decreasing diagrams. Yet, confluence by tiling may also be valid there, analogous to the case of braids. The well-known braid presentation does also not admit for its ‘canonical’ tiles an application of decreasing diagrams. Nevertheless confluence by tiling does hold there for its canonical tiles, and it can be proven using decreasing diagrams employing a more complex labelling [9].

[3] Already two labels suffice for proving confluence using decreasing diagrams of every countable abstract reduction system [10]. The completeness for uncountable systems remains a long-standing open problem [23]. For proving commutation (a generalisation of confluence involving two relations) decreasing diagrams are not complete, as established in [8].

Related work

Cassaigne, Espie, Krob, Novelli and Hivert [4] study combinatorial properties of the Chinese monoid. In particular, they determine the size and structure of the convertibility classes. Karpuz [14] gives a complete rewriting system for convertibility, obtained by critical pair completion.

2 Reduction diagrams for monoids

In the theory of abstract rewriting systems (ARSs) we often apply the technique of constructing reduction diagrams by gluing together 'elementary diagrams' (e.d.'s) or 'tiles' to obtain a finite, completed reduction diagram of which the convergent sides yield the desired confluent reductions. Also in the theory of braids and Garside monoids this is an important tool, there called 'word reversing', see Dehornoy [7].

As a preparation for the main section of this note where we prove confluence for the Chinese monoid C_n on n generators, we introduce this method, somewhat informally, guided by a few examples of monoids. More complete expositions of reduction diagram construction can be found in [22, 1, 9]. In the latter paper the exposition is for positive braid words.

Example 1. Consider the monoid with two generators $1, 2$ and relations

$$121 = 211 \qquad\qquad 221 = 212$$

This is actually the Chinese monoid C_2 on two generators, which coincides with the Plactic monoid P_2 on two generators. In Sections 4 and 5 we will consider these monoids C_n and P_n in the general case with n generators.

Suppose we are interested in the *confluence* question for this monoid:

$$\forall u, v.\ \exists x, y.\ \ ux = vy\ \ ?$$

where u, v, x, y are elements of C_2, and '=' is the monoid equality. Actually, we work with u, v, x, y as words in $\{1, 2\}^*$, subject to the equality generated by the two relations above. In this way we are dealing with *string rewriting*, see Book & Otto [2]. To address the confluence question, we now invoke the technique of constructing reduction diagrams.

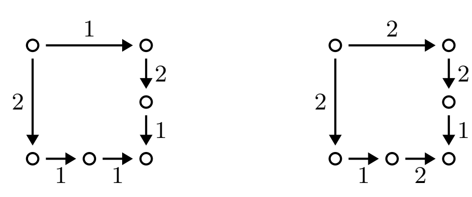


Fig. 1. Tiles for C_2 and P_2.

The relations 121 = 211 and 221 = 212 can be rendered as the elementary diagrams (e.d.'s) shown in Figure 1. Of these two basic tiles we have an infinite supply of copies, which moreover are scalable, horizontally and vertically.

We can now address the question for common extensions x, y for e.g. $u = 12$ and $v = 222$ by constructing the diagram D_0 in Figure 2.

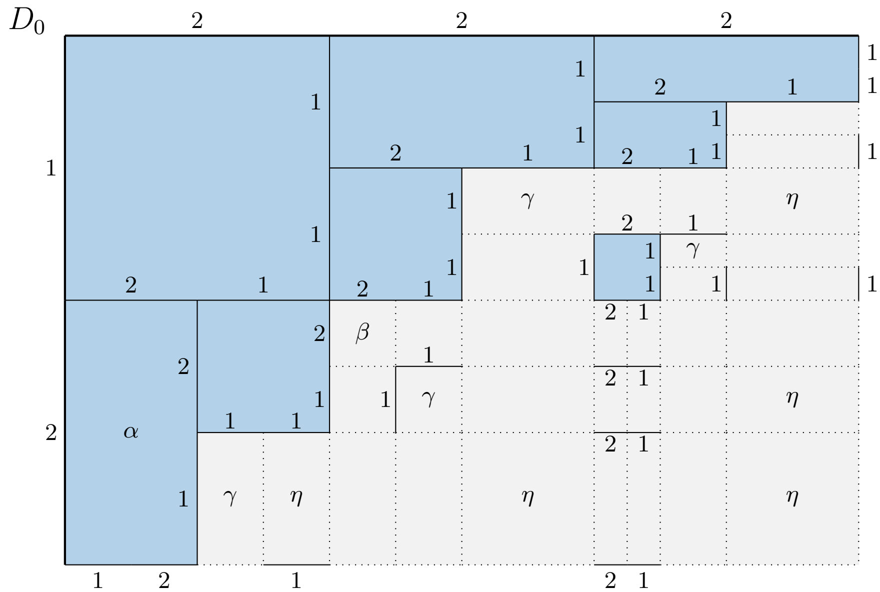


Fig. 2. Completed reduction diagram in C_2. Proper tiles are blue, trivial tiles involving empty steps are grey.

The result of this diagram construction is

$$12\ 12121 = 222\ 1111$$

In fact, the completed diagram contains also the actual conversion between 1212121 and 2221111:

$$
\begin{aligned}
&22\underline{211}11 = \\
&221\underline{211}1 = \\
&2211\underline{211} = \\
&2\underline{211}121 = \\
&21\underline{211}21 = \\
&\underline{211}2121 = \\
&1212121
\end{aligned}
$$

This can be seen by traversing the diagram from the right-upper corner and flipping repeatedly an elementary tile. We suppressed the conversion steps corresponding to the trivial tiles with their empty sides; these steps would involve the unit element ϵ of the monoid and equations such as $1\epsilon = \epsilon 1$ and $\epsilon\epsilon = \epsilon\epsilon$.

This example diagram evokes some clarifying comments:

(i) Note that for every 'open corner' or 'peak' in a stage of the construction we have a matching tile, for 1-against-1, 1-against-2, 2-against-2, 1-against-ϵ, 2-against-ϵ and ϵ-against-ϵ. Such a set of tiles is called *full*.

(ii) Tiles γ, of 1-against-1, are called *absorption tiles*. Also tile β, 2-against-2, is an absorption tile.

(iii) The set of tiles is *non-deterministic*: for an open corner 2-against-2 there are two choices to glue, the absorption tile β, or the tile α. Both are used in the construction of the diagram D_0.

(iv) Note the role of the 'degenerate' or 'trivial' tiles η, involving 2 or 4 empty steps ϵ. They serve to propagate steps to the right and downwards, and to keep the diagram in orthogonal shape.

(v) D_0 is a *completed* reduction diagram; no open corners i-against-j are left open.

The next example exhibits an infinite, cyclic reduction diagram.

Example 2. Consider the monoid

$$\langle a, b \mid ab = bba \rangle$$

Here we have the single elementary diagram, apart from the trivial ones, as follows (see Figure 3). This gives rise to the infinite cyclic reduction diagram, which is a well-known counterexample in abstract rewriting. It shows that in Newman's Lemma, mentioned above, see also [13], the condition SN cannot be missed.

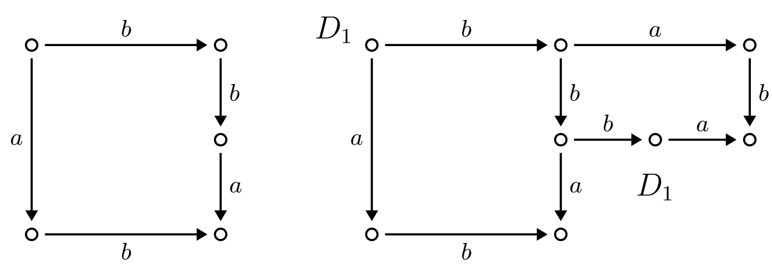


Fig. 3. Cyclic reduction diagram.

Example 3. A similar cyclic diagram construction (see Figure 4) is exhibited by the monoid

$$\langle a, b \mid aba = bba \rangle$$

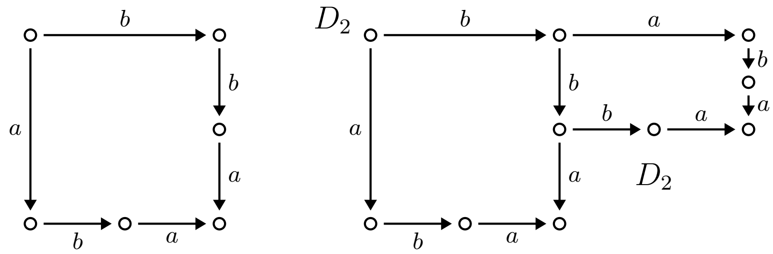


Fig. 4. Another cyclic reduction diagram.

Example 4. An interesting cyclic diagram construction (see Figure 5) is found in the Artin-Tits monoid, as mentioned in [7]:

$$\langle 1, 2, 3 \mid 121 = 212,\ 232 = 323,\ 131, 313 \rangle$$

In fact, this monoid describes braids on three strands that are placed on a cylinder.

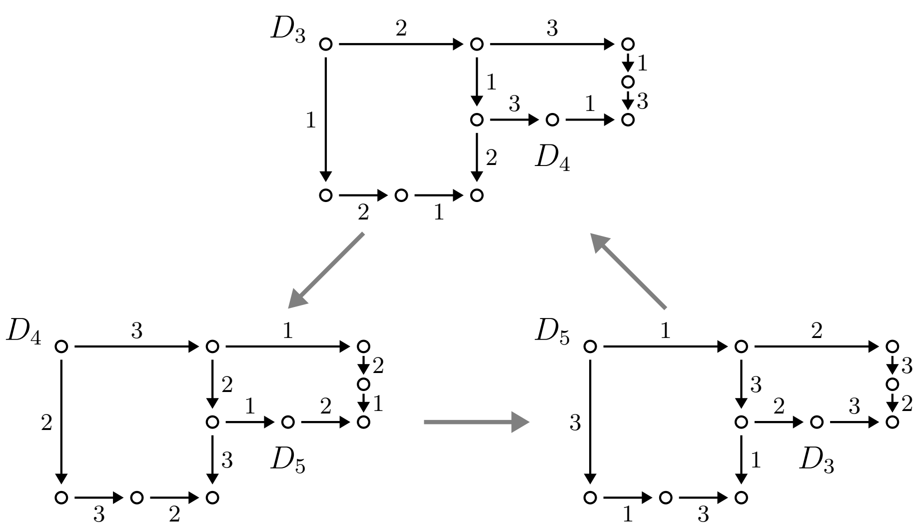


Fig. 5. Cyclic diagram in the Artin-Tits monoid.

Example 5. Consider the monoid

$$\langle 1, 2 \mid 121 = 211,\ 121 = 221 \rangle$$

This presentation with its two elementary diagrams (in addition to the trivial ones) exhibits the phenomenon that already a proper part of the set of elementary diagrams may be sufficient for successful reduction diagram completion, while the whole set may admit an unsuccessful, diverging diagram construction, by yielding a cyclic diagram or an otherwise infinite diagram.

For the present example the left e.d., together with the two absorption e.d.'s for a-against-a and b-against-b is sufficient; as we will see later this is so because it is a decreasing diagram. But the e.d. on the right may lead to infinite, cyclic diagrams as witnessed by Example 3.

This leads us to define:

Definition 6. Let $\mathcal{T}$ be a set of elementary diagrams:

1. $\mathcal{T}$ is called *sufficient for confluence by tiling*, for short *sufficient*, if for every pair of finite reductions σ set against τ, *some* gluing sequence leads to a completed reduction diagram $D(\sigma, \tau)$ using tiles from $\mathcal{T}$.
2. $\mathcal{T}$ is called *strongly sufficient for confluence by tiling*, for short *strongly sufficient*, if for every pair of finite reductions σ set against τ, *every* glueing sequence leads eventually to a completed reduction diagram $D(\sigma, \tau)$ using tiles from $\mathcal{T}$.

3 Decreasing diagrams

In the preceding section we have seen that, when we are lucky, confluence can be obtained by tiling, that is, the construction of completed reduction diagrams by repeatedly gluing a tile to a partially completed reduction diagram. However, sometimes this procedure fails. The process of tiling can be infinite without ever completing the reduction diagram, as the cyclic diagrams showed.

The decreasing diagrams technique ([6, 23, 22, 8, 10]) is one of the strongest techniques for guaranteeing that tiling will succeed (terminate and yield a completed reduction diagram). To guarantee termination of tiling, the technique employs a labelling on the steps (together with a well-founded ordering $<$ on the label set I), and it restricts the choice of tiles for gluing to 'decreasing elementary diagrams' as shown in Figure 6.

Definition 7 (Indexed abstract reduction system). *An* indexed abstract reduction system (ARS) $\mathcal{A} = \langle A, \{\rightarrow_i\}_{i\in I}\rangle$ *consists of a set of objects A, an index set I, and a binary relation $\rightarrow_i \subseteq A \times A$ for every $i \in I$. We write $\rightarrow$ for the union $\bigcup_{i\in I} \rightarrow_i$ of all the reduction relations.*

Notation 8 *For $< \;\subseteq I \times I$, we define $\rightarrow_{<\alpha} = \bigcup_{\beta<\alpha} \rightarrow_\beta$ and moreover $\rightarrow_{<\alpha\cup<\beta} = \rightarrow_{<\alpha} \cup \rightarrow_{<\beta}$.*

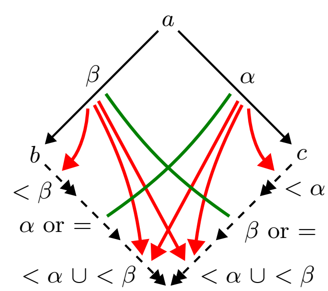


Fig. 6. The red arrows and green lines inside the diagram are intended as a visual aid. The red arrows stand for a strict decrease of the labels (multiple incoming arrows signify choice) while the green lines indicate a label carrying over unchanged. The double-headed arrows $\twoheadrightarrow$ are the transitive-reflexive closure of the one-step reduction relation $\rightarrow$.

Definition 9 (Decreasing elementary diagrams). *Let $\mathcal{A} = \langle A, \{\rightarrow_i\}_{i \in I}\rangle$ be an ARS and $< \subseteq I \times I$ a well-founded partial order. A reduction diagram of the form shown in Figure 6 is called a* decreasing elementary diagram *for the peak $c \leftarrow_\beta a \rightarrow_\alpha b$.*

Theorem 10 (Decreasing diagrams [6, 23]). *Let $\mathcal{A} = \langle A, \{\rightarrow_i\}_{i \in I}\rangle$ be an ARS and $< \subseteq I \times I$ a well-founded partial order. Let T be a set of decreasing elementary diagrams that contains at least one elementary diagram for each peak in $\mathcal{A}$. Then T is strongly sufficient for confluence by tiling of $\mathcal{A}$.*

Example 11. The following is a list of the relations between ternary words over the alphabet $\{1, 2\}$ whose corresponding elementary diagrams are decreasing, with respect to the ordering $1 < 2$:

$211 = 111$	$121 = 211$	$212 = 212$
$212 = 211$	$221 = 211$	$221 = 221$
$221 = 212$	$212 = 112$	

Note the last two symmetrical equations with identical sides; in a presentation they would be useless, but they do give rise to decreasing tiles.

Also note the sensitivity with respect to the chosen ordering: for the ordering $2 < 1$ none of the tiles corresponding to the equations as listed below would be decreasing.

Finally, note that equations $211 = 112$ and $121 = 221$, obtained from the listed ones by transitivity, are not decreasing (for $1 < 2$).

Example 12. Earlier we considered the presentation of the monoid

$$\langle a, b \mid ab = bba\rangle$$

called in [7, p.73, Example 4.28], the Baumslag-Solitar monoid, having a cyclic diverging diagram for its canonical tiles, which is in different notation also included there. The corresponding tile, left in Figure 4, is therefore not decreasing, for no order on $\{ a, b \}$.

A *caveat* may be in order here: at other places in the literature Baumslag-Solitar monoids are said to have the presentation

$$\langle a, b \mid ab = ba^k \rangle$$

Note that the latter presentations for all k, do *not* have a diverging reduction diagram, as the canonical tiles are examples of decreasing diagrams, for $b > a$. (For $a > b$ they are not decreasing, for $k > 1$.)

4 Confluence of the Chinese monoid

Let Σ be an alphabet equipped with a total order $<$. Let $= \subseteq \Sigma^* \times \Sigma^*$ be the congruence generated by

$$aba = baa,\ bba = bab$$

for every $a < b$, and

$$cab = cba = bca$$

for every $a < b < c$.

So, '=' is an equivalence relation (reflexive, symmetric and transitive) and $lxr = lyr$ whenever $x = y$ is one of the equations above and $l, r \in \Sigma^*$.

Notation 13 For a word $x \in \Sigma^*$, we write $x^=$ for $\{ y \in \Sigma^* \mid x = y \}$, the equivalence class of x. For a set $X \subseteq \Sigma^*$ of words, let $X^= = \{ x^= \mid x \in X \}$.

The (right) word extension in the Chinese monoid can be viewed as an abstract reduction system as follows.

Definition 14. *The* (right) word extension ARS C *for the Chinese monoid over* $\langle \Sigma, < \rangle$ *is*

$$C = \langle \Sigma^{*=}, \{\to_i\}_{i \in \Sigma} \rangle$$

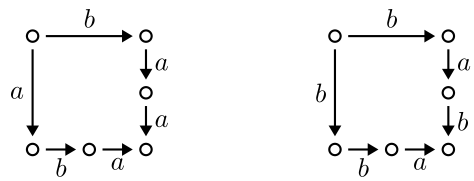
where, for every $i \in \Sigma$*, the relation* $\to_i$ *is defined by*

$$w^= \to_i (wi)^=$$

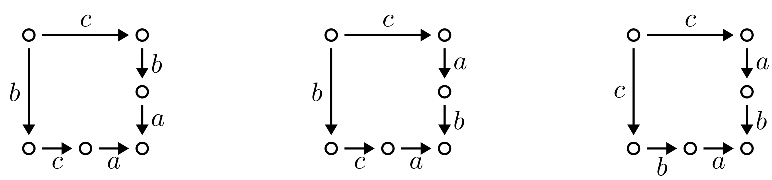
for every $w \in \Sigma^*$*.*

The ARS C has elementary diagrams as given in Definition 15. These diagrams arise naturally from the defining equations above of the Chinese monoid.

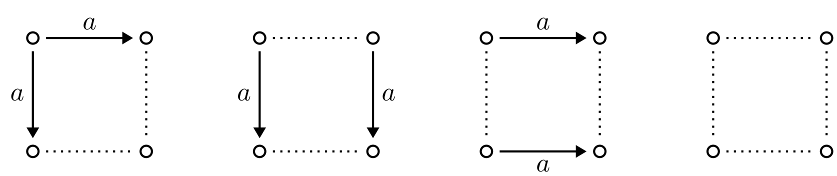
Definition 15 (Elementary diagrams for the Chinese monoid). *For the Chinese monoid over a totally ordered alphabet, we have the following elementary diagrams*



for every $a < b$, and



for every $a < b < c$, and trivial elementary diagrams



for every a. Here the dotted lines without arrowhead stand for empty steps.

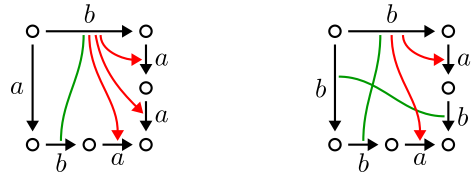
The application of decreasing diagrams for proving confluence typically requires

(a) a careful choice of the labelling of the steps, and
(b) a careful choice of the elementary diagrams (if there are multiple ways to join a peak).

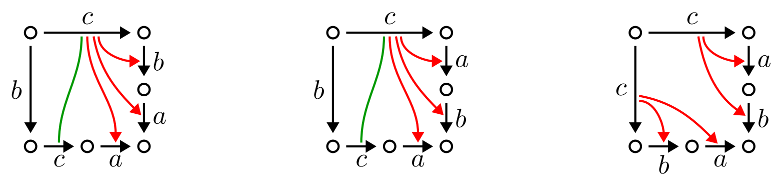
Somewhat surprisingly, for the Chinese monoid, neither of these steps is necessary. It turns out that the natural labelling of the steps with letters from Σ suffices to make all elementary diagrams in Definition 15 decreasing.

Proposition 16. *All elementary diagrams in Definition 15 are decreasing elementary diagrams with respect to the order $<$ on Σ.*

Proof. The trivial elementary diagrams are decreasing for every ordering on the labels. So it suffices to consider the non-trivial elementary diagrams. These are:



for every $a < b$, and



for every $a < b < c$.

All five of these configurations are allowed by the shape of decreasing elementary diagrams shown in Figure 6. Just like in Figure 6 we have used red arrows and green lines inside the elementary diagrams as a visual aid. The red arrows indicate a strict decrease of the label, while the green lines signify a label carrying over unchanged.

Theorem 17. *The set of elementary diagrams in Definition 15 is strongly sufficient for confluence by tiling for C (i.e., for confluence of right word extension in the Chinese monoid).*

Proof. By Proposition 16 all elementary diagrams in Definition 15 are decreasing. Moreover, this set of elementary diagrams is exhaustive in the sense that, for every peak in C, there exists a matching elementary diagram: it is a full set of tiles. Thus Theorem 10 is applicable.

Actually, the two diagrams for $a < b$ in Figure 7 would suffice for establishing confluence. However, the two corresponding equations are not suitable for the intended monoid presentation, as they do not generate the whole equality.

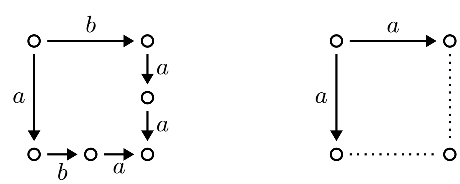


Fig. 7. Subset of tiles sufficient for confluence by tiling for C_n.

The point of Theorem 17 is not merely the confluence property. The crucial observation is that tiling of reduction diagrams always succeeds with the tiles from Definition 15 independent of the gluing strategy. As all these diagrams are decreasing, tiling is always guaranteed to terminate.

5 Confluence of the Plactic monoid

In the last section we have established confluence for the Chinese monoid C_n. We have as a corollary also the confluence of the Plactic monoid P_n, somewhat trivially. We show this for the case $n = 3$; the general case then follows easily. Consider the table in Figure 8. For the Chinese monoid C_3 we have the 8 relations in the left column which gives (together with the trivial tiles involving 2 or 4 empty sides) a nondeterministic, full set of tiles, all of which are decreasing (blue). The grouping shows the nondeterministic choices that are possible when gluing the tiles together towards a completed diagram.

$\underline{211 = 121}$	$\underline{211 = 121}$	$(\star)$
	$231 = 213$	
$\underline{221 = 212}$	$\underline{221 = 212}$	
$321 = 231$		
$312 = 231$		
$\underline{322 = 232}$	$\underline{322 = 232}$	$(\star)$
	$233 = 323$	
$\underline{311 = 131}$	$\underline{311 = 131}$	$(\star)$
	$132 = 312$	
$\underline{332 = 323}$	$\underline{332 = 323}$	
$\underline{331 = 313}$	$\underline{331 = 313}$	

Fig. 8. Relations of C_3 (left) and P_3 (right). Blue is decreasing for the natural ordering, red non-decreasing. Common relations are underlined. The three starred ones are already sufficient for confluence of P_3, they correspond to the two tiles in Figure 7, but do not generate the whole equality of P_3.

For the Plactic monoid P_3 we have the 9 relations as in the second column of Figure 8, where blue is decreasing and red non-decreasing.

The 6 underlined equations in both columns are the intersection between the ones of C_3 and P_3. This intersection corresponds to a full set of decreasing tiles, also nondeterministic. It follows that this set is also for P_3 a sufficient set of tiles. Hence also P_3 is confluent.

The same remark as above for C_n applies: already the three $(\star)$ equations shared with C_n together with the two absorption tiles for 1-against-1 and 2-against-2 are sufficient for confluence. But note that this trio of equations does not generate the whole equality in C_3, nor in P_3.

An interesting question is whether the set of tiles for P_3 is also *strongly* sufficient. We have not been able to find a diverging reduction diagram for P_3; we conjecture that it does not exist, so that the set of tiles is strongly sufficient.

6 Conclusions and further questions

1. We have applied the technique of constructing reduction diagrams, confluence by tiling, and decreasing diagrams to the important Chinese and Plactic monoid, obtaining confluence in the sense of having common extensions of elements of these monoids, in analogy to the braids monoid. For braids there is moreover an equivalence between convertibility of braid words and Lévy's *projection equivalence*. For C_n and P_n this is more complicated because their sets of tiles are nondeterministic, admitting choices in the diagram completion. Therefore the notion of 'projection' is not defined unequivocally.
2. A question is whether the full set of tiles for the Plactic monoid is also strongly sufficient, as is the case for the Chinese monoid.
3. An important question is how the various notions such as confluence, confluence by tiling are dependent on the actual presentation of monoids. These presentations can be varied by applying Tietze moves. Some notions are known to be 'absolute' in this respect, they hold for every presentation, and thus are properties of the monoid and not merely of the monoid presentation. Having *finite derivation type* is such an important absolute property (see [15].) We expect that confluence is also an absolute property. For confluence by tiling the absoluteness is also an interesting question.
4. For the braids monoid an interesting fact is that the usual presentation can be via Tietze moves transformed to consist of *short* relations, sometimes called *Ore-conditions*. These are of the form $a = bc$ or $ab = cd$. The corresponding tiles are then *non-splitting*, i.e. have converging sides consisting of a single step or an empty step ϵ. Confluence by tiling is then trivial, as all tiles are simple squares.

   We wonder whether also the Chinese and Plactic monoid possess such a presentation with only short relations.
5. Another question is whether the confluence property for monoid presentations is *decidable*; and the same for the stronger property of confluence by tiling.
6. The infinite diagrams arising from some monoid presentations such as the Baumslag-Solitar monoid in Figure 3 and the Artin-Tits monoid in Figure 5 are intriguing objects themselves. The examples of infinite reduction diagrams seen above are all *cyclic*, involving a proper copy of themselves. Two questions arise: are there also *non-cyclic infinite reduction diagrams* arising from the tiles of monoids?

   A second question concerns the *finite and infinite traces* that arise in infinite diagrams starting at the root of the diagram, the left-upper corner. Are the sets of infinite traces arising from infinite diagrams for monoid presentations, as seems to be the case in the examples considered above, always ω*-regular languages*? We offer this puzzle happily to Catuscia!

Applications of confluence by tiling, possibly combined with an application of decreasing diagrams, certainly does not stop with the Chinese monoid. Various other monoids can be subjected to an analysis with these two tools, confluence by tiling and decreasing diagrams. We hope to demonstrate this in subsequent work.

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