Eigenvalues and Transduction of Morphic Sequences

David Sprunger**, William Tune**, Jörg Endrullis*, and Lawrence S. Moss**

Abstract. We study finite state transduction of automatic and morphic sequences. Dekking [4] proved that morphic sequences are closed under transduction and in particular morphic images. We present a simple proof of this fact, and use the construction in the proof to show that non-erasing transductions preserve a condition called α-substitutivity. Roughly, a sequence is α-substitutive if the sequence can be obtained as the limit of iterating a substitution with dominant eigenvalue α. Our results culminate in the following fact: for multiplicatively independent real numbers α and β, if v is a α-substitutive sequence and w is an β-substitutive sequence, then v and w have no common non-erasing transducts except for the ultimately periodic sequences. We rely on Cobham's theorem for substitutions, a recent result of Durand [5].

1 Introduction

Infinite sequences of symbols are of paramount importance in a wide range of fields, ranging from formal languages to pure mathematics and physics. A landmark was the discovery in 1912 by Axel Thue, founding father of formal language theory, of the famous sequence 0110 1001 1001 0110 1001 0110 $\cdots$.Thue was interested in infinite words which avoid certain patterns, like squares ww or cubes www, when w is a non-empty word. Indeed, the sequence shown above, called the Thue–Morse sequence, is cube-free. It is perhaps the most natural cube-free infinite word.

A common way to transform infinite sequences is by using *finite state transducers*. These transducers are deterministic finite automata with input letters and output words for each transition; an example is shown in Figure 1. Usually we omit the words "finite state" and refer to *transducers*. A transducer maps infinite sequences to infinite sequences by reading the input sequence letter by letter. Each of these transitions produces an output word, and the sequence formed by concatenating each of these output words in the order they were produced is the output sequence. In particular, since this

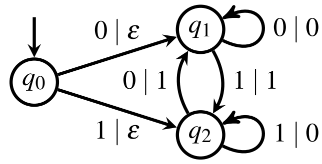

Fig. 1. A transducer computing the difference (exclusive or) of consecutive bits.

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transducer runs for infinite time to read its entire input, this model of transduction does not have final states. A transducer is called *k-uniform* if each step produces k-letter words. For example, *Mealy machines* are 1-uniform transducers. A transducer is *non-erasing* if each step produces a non-empty word; this condition is prominent in this paper.

Although transducers are a natural machine model, hardly anything is known about their capabilities of transforming infinite sequences. To state the issues more clearly, let us write $x \trianglelefteq y$ if there is a transducer taking y to x. This transducibility gives rise to a partial order of *stream degrees* [6] that is analogous to, but more fine-grained than, recursion-theoretic orderings such as *Turing reducibility* $\leq_T$ and *many-one reducibility* $\leq_m$. We find it surprising that so little is known about $\trianglelefteq$. As of now, the structure of this order is vastly unexplored territory with many open questions. To answer these questions, we need a better understanding of transducers.

The main things that are known at this point concern two particularly well-known sets of streams, namely the *morphic* and *automatic* sequences. Morphic sequences are obtained as the limit of iterating a morphism on a starting word (and perhaps applying a coding to the limit word). Automatic sequences have a number of independent characterizations (see [1]); we shall not repeat these here. There are two seminal closure results concerning the transduction of morphic and automatic sequences:

(1) The class of morphic sequences is closed under transduction (Dekking [4]).
(2) For all k, the class of k-automatic sequences is closed under uniform transduction (Cobham [3]).

In this paper, we do not attack the central problems concerning the stream degrees. Instead, we are interested in a closure result for non-erasing transductions. Our interest comes from the following easy observation:

(3) For every morphic sequence $w \in \Sigma^\omega$ there is a 2-automatic sequence $w' \in (\Sigma \cup \{ a \})^\omega$ such that w is obtained from w' by erasing all occurrences of a. (See Allouche and Shallit [1, Theorem 7.7.1])

This motivates the question: how powerful is non-erasing transduction?

Our contribution. The main result of this paper is stated in terms of the notion of *α-substitutivity*. This condition is defined in Definition 5 below, and the definition uses the eigenvalues of matrices naturally associated with morphisms on finite alphabets. Indeed, the core of our work is a collection of results on eigenvalues of these matrices.

We prove that the set of α-substitutive words is closed under non-erasing finite state transduction. We follow Allouche and Shallit [1] in obtaining transducts of a given morphic sequence w by *annotating* an iteration morphism, and then taking a morphic image of the annotated limit sequence. For the first part of this transformation, we show that a morphism and its annotation have the same eigenvalues with non-negative eigenvectors. For the second part, we revisit the proof given in Allouche and Shallit [1] of Dekking's theorem that morphic images of morphic sequences are morphic. We simplify the construction in the proof to make it amenable for an analysis of the eigenvalues of the resulting morphism.

For an extended version of this paper with examples we refer to [9].

Related work. Durand [5] proved that if w is an α-substitutive sequence and h is a non-erasing morphism, then $h(w)$ is α^k-substitutive for some $k \in \mathbb{N}$. We strengthen this result in two directions. First, we show that k may be taken to be 1; hence $h(w)$ is α^k-substitutive for every $k \in \mathbb{N}$. Second, we show that Durand's result also holds for non-erasing transductions.

2 Preliminaries

We recall some of the main concepts that we use in the paper. For a thorough introduction to morphic sequences, automatic sequences and finite state transducers, we refer to [1,8].

We are concerned with infinite sequences Σ^ω over a finite alphabet Σ. We write Σ^* for the set of finite words, Σ^+ for the finite, non-empty words, Σ^ω for the infinite words, and $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ for all finite or infinite words over Σ.

2.1 Morphic sequences and automatic sequences

Definition 1. *A* morphism *is a map $h : \Sigma \to \Gamma^*$. This map extends by concatenation to $h : \Sigma^* \to \Gamma^*$, and we do not distinguish the two notationally. Notice also that $h(vu) = h(v)h(u)$ for all $u, v \in \Sigma^*$. If $h_1, h_2 : \Sigma \to \Sigma^*$, we have a composition $h_2 \circ h_1 : \Sigma \to \Sigma^*$.*

An erased letter *(with respect to h) is some $a \in \Sigma$ such that $h(a) = \varepsilon$. A morphism $h : \Sigma^* \to \Gamma^*$ is called* erasing *if has an erased letter. A morphism is k-*uniform *(for $k \in \mathbb{N}$) if $|h(a)| = k$ for all $a \in \Sigma$. A* coding *is a 1-uniform morphism $c : \Sigma \to \Gamma$.*

A morphic sequence is obtained by iterating a morphism, and applying a coding to the limit word.

Definition 2. *Let $s \in \Sigma^+$ be a word, $h : \Sigma \to \Sigma^*$ a morphism, and $c : \Sigma \to \Gamma$ a coding. If the limit $h^\omega(s) = \lim_{n\to\infty} h^n(s)$ exists and is infinite, then $h^\omega(s)$ is a* pure morphic sequence, *and $c(h^\omega(s))$ a* morphic sequence.

If $h(x_1) = x_1 z$ for some $z \in \Sigma^+$, then we say that h is prolongable *on x_1. In this case, $h^\omega(x_1)$ is a pure morphic sequence.*

*If additionally, the morphism h is k-uniform, then $c(h^\omega(s))$ is a k-*automatic sequence. *A sequence $w \in \Sigma^\omega$ is called* automatic *if w is k-automatic for some $k \in \mathbb{N}$.*

2.2 Cobham's Theorem for morphic words

Definition 3. *For $a \in \Sigma$ and $w \in \Sigma^*$ we write $|w|_a$ for the* number of occurrences of *a* in *w*. *Let h be a morphism over Σ. The* incidence matrix *of h is the matrix $M_h = (m_{i,j})_{i\in\Sigma, j\in\Sigma}$ where $m_{i,j} = |h(j)|_i$ is the number of occurrences of the letter i in the word $h(j)$.*

Theorem 4 (Perron-Frobenius). *Every non-negative square matrix M has a real eigenvalue $\alpha \geq 0$ that is greater than or equal to the absolute value of any other eigenvalue of M and the corresponding eigenvector is non-negative. We refer to α as the* dominating eigenvalue *of* M*.*

Definition 5. *The* dominating eigenvalue *of a morphism* h *is the dominating eigenvalue of* M_h*. An infinite sequence* $w \in \Sigma^\omega$ *over a finite alphabet* Σ *is said to be* α-substitutive *(*$\alpha \in \mathbb{R}$*) if there exist a morphism* $h : \Sigma \to \Sigma^*$ *with dominating eigenvalue* α*, a coding* $c : \Sigma \to \Sigma$ *and a letter* $a \in \Sigma$ *such that (i)* $w = c(h^\omega(a))$*, and (ii) every letter of* Σ *occurs in* $h^\omega(a)$*.*

Two complex numbers x, y are called *multiplicatively independent* if for all $k, \ell \in \mathbb{Z}$ it holds that $x^k = y^\ell$ implies $k = \ell = 0$. We shall use the following version of Cobham's theorem due to Durand [5].

Theorem 6. *Let* α *and* β *be multiplicatively independent Perron numbers. If a sequence* w *is both* α*-substitutive and* β*-substitutive, then* w *is eventually periodic.* □

2.3 Transducers

Definition 7. *A* (sequential finite-state) transducer (FST) $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ *consists of (i) a finite input alphabet* Σ*, (ii) a finite output alphabet* Δ*, (iii) a finite set of states* Q*, (iv) an initial state* $q_0 \in Q$*, (v) a transition function* $\delta : Q \times \Sigma \to Q$*, and (vi) an output function* $\lambda : Q \times \Sigma \to \Delta^*$*.*

We use transducers to transform infinite words. The transducer reads the input word letter by letter, and the transformation result is the concatenation of the output words encountered along the edges.

Definition 8. Let $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ be a transducer. We extend the state transition function δ from letters Σ to finite words Σ^* as follows: $\delta(q, \varepsilon) = q$ and $\delta(q, aw) = \delta(\delta(q, a), w)$ for $q \in Q$, $a \in \Sigma$, $w \in \Sigma^*$.

The output function λ is extended to the set of all words $\Sigma^\infty = \Sigma^\omega \cup \Sigma^*$ by the following definition: $\lambda(q, \varepsilon) = \varepsilon$ and $\lambda(q, aw) = \lambda(q, a)\,\lambda(\delta(q, a), w)$ for $q \in Q$, $a \in \Sigma$, $w \in \Sigma^\infty$.

We introduce $\delta(w)$ and $\lambda(w)$ as shorthand for $\delta(q_0, w)$ and $\lambda(q_0, w)$, respectively. Moreover, we define $M(w) = \lambda(w)$, the *output of* M *on* $w \in \Sigma^\omega$. In this way, we think of M as a function from (finite or infinite) words on its input alphabet to infinite words on its output alphabet $M : \Sigma^\infty \to \Delta^\infty$. If $x \in \Sigma^\omega$ and $y \in \Delta^\omega$, we write $y \trianglelefteq x$ if for some transducer M, we have $M(x) = y$.

Notice that every morphism is computable by a transducer (with one state). In particular, every coding is computable by a transducer.

3 Closure of Morphic Sequences under Morphic Images

Definition 9. Let $h : \Sigma^* \to \Sigma^*$ be morphisms, and let $\Gamma \subseteq \Sigma$ be a set of letters. We call a letter $a \in \Sigma$

(i) *dead* if $h^n(a) \in \Gamma^*$ for all $n \geq 0$,
(ii) *near dead* if $a \notin \Gamma$, and for all $n > 0$, $h^n(a)$ consists of dead letters,
(iii) *resilient* if $h^n(a) \notin \Gamma^*$ for all $n \geq 0$,
(iv) *resurrecting* if $a \in \Gamma$ and $h^n(a) \notin \Gamma^*$ for all $n > 0$

with respect to h and Γ. We say that the morphism h *respects* Γ if every letter

$a \in \Sigma$ is either dead, near dead, resilient, or resurrecting. (Note that all of these definitions are with respect to some fixed h and Γ.)

Lemma 10. *Let $g : \Sigma^* \to \Sigma^*$ be a morphism, and let $\Gamma \subseteq \Sigma$. Then g^r respects Γ for some natural number $r > 0$.*

Proof. See Lemma 7.7.3 in Allouche and Shallit [1]. □

Definition 11. *For a set of letters $\Gamma \subseteq \Sigma$ and a word $w \in \Sigma^\infty$, we write $\gamma_\Gamma(w)$ for the word obtained from w by erasing all occurrences of letters in Γ.*

Definition 12. Let $g : \Sigma^* \to \Sigma^*$ be a morphism, and $\Gamma \subseteq \Sigma$ a set of letters. We construct an alphabet Δ, a morphism $\xi : \Delta^* \to \Delta^*$ and a coding $\rho : \Delta \to \Sigma$ as follows. We refer to Δ, ξ, ρ as the *morphic system associated with the erasure of Γ from g^ω*.

Let $r \in \mathbb{N}_{>0}$ be minimal such that g^r respects Γ (r exists by Lemma 10). Let $\mathcal{D}$ be the set of dead letters with respect to g^r and Γ. For $x \in \Sigma^*$ we use brackets $[x]$ to denote a new letter. For words $w \in \{g^r(a) \mid a \in \Sigma\}$, whenever $\gamma_{\mathcal{D}}(w) = w_0\ a_1w_1\ a_2w_2\ \cdots\ a_{k-1}w_{k-1}\ a_kw_k$ with $a_1, \ldots, a_k \notin \Gamma$ and $w_0, \ldots, w_k \in \Gamma^*$, we define $\mathsf{blocks}(w) = [w_0a_1w_1]\ [a_2w_2]\ \cdots\ [a_{k-1}w_{k-1}]\ [a_kw_k]$. Here it is to be understood that $\mathsf{blocks}(w) = \varepsilon$ if $\gamma_{\mathcal{D}}(w) = \varepsilon$, and $\mathsf{blocks}(w)$ is *undefined* if $\gamma_{\mathcal{D}}(w) \in \Gamma^+$.

Let the alphabet Δ consist of all letters $[a]$ and all bracketed letters $[w]$ occurring in words $\mathsf{blocks}(g^r(a))$ for $a \in \Sigma$. We define the morphism $\xi : \Delta \to \Delta^*$ and the coding $\rho : \Delta \to \Sigma$ by $\xi([a_1 \cdots a_k]) = \mathsf{blocks}(g^r(a_1)) \cdots \mathsf{blocks}(g^r(a_k))$ and $\rho([w\,a\,u]) = a$ for $[a_1 \cdots a_k] \in \Delta$ and $a \notin \Gamma$, $w, u \in \Gamma^*$. For $a \in \Gamma$ we can define $\rho([a])$ arbitrarily, for example, $\rho(a) = a$.

Proposition 13. *Let $g : \Sigma^* \to \Sigma^*$ be a morphism, $a \in \Sigma$ such that $g^\omega(a) \in \Sigma^\omega$, and $\Gamma \subseteq \Sigma$ a set of letters. Let Δ, ξ and ρ be the morphic system associated to the erasure of Γ from g^ω in Definition 12. Then $\rho(\xi^\omega([a])) = \gamma_\Gamma(g^\omega(a))$.*

Proof. For $\ell \in \mathbb{N}$ and $[w_1], \ldots, [w_\ell] \in \Delta$ we define $\mathsf{cat}([w_1] \cdots [w_\ell]) = w_1 \cdots w_\ell$. We prove by induction on n that for all words $w \in \Delta^*$, and for all $n \in \mathbb{N}$, $\mathsf{cat}(\xi^n(w)) = g^{nr}(\mathsf{cat}(w))$. The base case is immediate. For the induction step, assume that we have $n \in \mathbb{N}$ such that for all words $w \in \Delta^*$, $\mathsf{cat}(\xi^n(w)) = g^{nr}(\mathsf{cat}(w))$. Let $w \in \Delta^*$, $w = [a_{1,1} \cdots a_{1,\ell_1}] \cdots [a_{k,1} \cdots a_{k,\ell_k}]$. Then

$$\begin{aligned}\mathsf{cat}(\xi(w)) &= \mathsf{cat}(\xi([a_{1,1} \cdots a_{1,\ell_1}])\ \cdots\ \xi([a_{k,1} \cdots a_{k,\ell_k}]))\\ &= \mathsf{cat}(\mathsf{blocks}(g^r(a_{1,1})) \cdots \mathsf{blocks}(g^r(a_{k,\ell_k}))) = g^r(\mathsf{cat}(w))\end{aligned}$$

By the induction hypothesis, $\mathsf{cat}(\xi^{n+1}(w)) = g^{nr}(\mathsf{cat}(\xi(w))) = g^{nr}(g^r(\mathsf{cat}(w))) = g^{(n+1)r}(\mathsf{cat}(w))$. To complete the proof, note that by definition $\rho([w\,a\,u]) = \gamma_\Gamma(w\,a\,u)$ and thus $\rho(w) = \gamma_\Gamma(\mathsf{cat}(w))$ for every $w \in \Delta^*$. Hence, for all $n \geq 1$, $\rho(\xi^n([a])) = \gamma_\Gamma(\mathsf{cat}(\xi^n([a]))) = \gamma_\Gamma(g^{nr}(a))$. Taking limits: $\rho(\xi^\omega([a])) = \gamma_\Gamma(g^\omega(a))$. □

Definition 14. Let $g, h : \Sigma^* \to \Sigma^*$ be morphisms such that h is non-erasing. We construct an alphabet Δ, a morphism $\xi : \Delta^* \to \Delta^*$ and a coding $\rho : \Delta \to \Sigma$ as follows. We refer to Δ, ξ, ρ as the *morphic system associated with the morphic image of g^ω under h*.

Let $\Delta = \Sigma \cup \{\,[a] \mid a \in \Sigma\,\}$. For nonempty words $w = a_1a_2 \cdots a_k \in \Sigma^*$ we define $\mathsf{head}(w) = a_1$, $\mathsf{tail}(w) = a_2 \cdots a_k$ and $\mathsf{img}(w) = [a_1]u_1\ [a_2]u_2\ \cdots\ [a_{k-1}]u_{k-1}\ [a_k]u_k$

where $u_i = \mathsf{tail}(h(a_i)) \in \Sigma^*$. We define the morphism $\xi : \Delta^* \to \Delta^*$ and the coding $\rho : \Delta \to \Sigma$ by $\xi([a]) = \mathsf{img}(g(a)))$ and $\xi(a) = \varepsilon$, and $\rho([a]) = \mathsf{head}(h(a))$ and $\rho(a) = a$ for $a \in \Sigma$.

Notice here the $\rho([a])$ and u_i, defined using $\mathsf{head}()$ and $\mathsf{tail}()$, are well-defined since h is non-erasing and hence $h(a_i)$ will be nonempty.

Proposition 15. *Let $g, h : \Sigma^* \to \Sigma^*$ be morphisms such that h is non-erasing, and $a \in \Sigma$ such that $g^\omega(a) \in \Sigma^\omega$. Let Δ, ξ and ρ be as in Definition 12. Then $\rho(\xi^\omega([a])) = h(g^\omega(a))$.*

Proof. We define $z : \Delta \to \Sigma^*$ by $z(a) = \varepsilon$ and $z([a]) = a$ for all $a \in \Sigma$. By induction on $n > 0$ we show $\rho(\xi^n(w)) = h(g^n(z(w)))$ and $z(\xi^n(w)) = g^n(z(w))$ for all $w \in \Delta^*$.

We start with the base case. Note that $\rho(\xi([a])) = h(g(a)) = h(g(z([a])))$ and $\rho(\xi(a)) = \varepsilon = h(g(z(a)))$ for all $a \in \Sigma$, and thus $\rho(\xi(w)) = h(g(z(w)))$ for all $w \in \Delta^*$. Moreover, we have $z(\xi([a])) = g(a) = g(z([a]))$ and $z(\xi(a)) = \varepsilon = g(z(a))$ for all $a \in \Sigma$, and hence $z(\xi(w)) = g(z(w))$ for all $w \in \Delta^*$.

Let us consider the induction step. By the base case and induction hypothesis $\rho(\xi^{n+1}(w)) = \rho(\xi(\xi^n(w))) = h(g(z(\xi^n(w)))) = h(g(g^n(z(w)))) = h(g^{n+1}(z(w)))$ and $z(\xi^{n+1}(w)) = z(\xi(\xi^n(w))) = g(z(\xi^n(w))) = g(g^n(z(w))) = g^{n+1}(z(w))$. Thus $\rho(\xi^n([a])) = h(g^n(a))$ for all $n \in \mathbb{N}$, and taking limits: $\rho(\xi^\omega([a])) = h(g^\omega(a))$. ◻

Every morphic image of a word can be obtained by erasing letters, followed by the application of a non-erasing morphism. As a consequence we obtain:

Corollary 16. *The morphic image of a pure morphic word is morphic or finite.*

Proof. Let $w \in \Sigma^\omega$ be a word and $h : \Sigma \to \Sigma^*$ a morphism. Let $\Gamma = \{\, a \mid h(a) = \varepsilon \,\}$ be the set of letters erased by h, and $\Delta = \Sigma \setminus \Gamma$. Then $h(w) = g(\gamma_\Gamma(w))$ where g is the non-erasing morphism obtained by restricting h to Δ. Hence for purely morphic w, the result follows from Propositions 13 and 15. ◻

Theorem 17 (Cobham [2], Pansiot [7]). *The morphic image of a morphic word is morphic.*

Proof. Follows from Corollary 16 since the coding can be absorbed into the morphic image. ◻

Eigenvalue analysis

The following lemma states that if a square matrix N is an extension of a square matrix M, and all added columns contain only zeros, then M and N have the same non-zero eigenvalues.

Lemma 18. *Let Σ, Δ be disjoint, finite alphabets. Let $M = (m_{i,j})_{i,j \in \Sigma}$ and $N = (n_{i,j})_{i,j \in \Sigma \cup \Delta}$ be matrices such that (i) $n_{i,j} = m_{i,j}$ for all $i, j \in \Sigma$ and (ii) $n_{i,j} = 0$ for all $i \in \Sigma \cup \Delta$, $j \in \Delta$. Then M and N have the same non-zero eigenvalues.*

$$\left(\begin{array}{c|c} M & 0 \cdots 0 \\ \hline & 0 \cdots 0 \\ & 0 \cdots 0 \end{array}\right)$$

Proof. N is a block lower triangular matrix with M and 0 as the matrices on the diagonal. Hence the eigenvalues of N are the combined eigenvalues of M and 0. Therefore M and N have the same non-zero eigenvalues. ◻

We now show that morphic images with respect to non-erasing morphisms preserve α-substitutivity. This strengthens a result obtained in [5] where it has been shown that the non-erasing morphic image of an α-substitutive sequence is α^k-substitutive for some $k \in \mathbb{N}$. We show that one can always take $k = 1$. Note that every α-substitutive sequence is also α^k-substitutive for all $k \in \mathbb{N}, k > 0$.

Theorem 19. *Let Σ be a finite alphabet, $w \in \Sigma^\omega$ be an α-substitutive sequence and $h : \Sigma \to \Sigma^*$ a non-erasing morphism. Then the morphic image of w under h, that is $h(w)$, is α-substitutive.*

Proof. Let $\Sigma = \{a_1, \ldots, a_k\}$ be a finite alphabet, $w \in \Sigma^\omega$ be an α-substitutive sequence and $h : \Sigma \to \Sigma^*$ a non-erasing morphism. As the sequence w is α-substitutive, there exist a morphism $g : \Sigma \to \Sigma^*$ with dominant eigenvalue α, a coding $c : \Sigma \to \Sigma$ and a letter $a \in \Sigma$ such that $w = c(g^\omega(a))$ and all letters from Σ occur in $g^\omega(a)$. Then $h(w) = h(c(g^\omega(a))) = (h \circ c)(g^\omega(a)))$, and $h \circ c$ is a non-erasing morphism. Without loss of generality, by absorbing c into h, we may assume that c is the identity.

From h and g, we obtain an alphabet Δ, a morphism ξ, and a coding ρ as in Definition 14. Then by Proposition 15, we have $\rho(\xi^\omega([a])) = h(g^\omega(a))$. As a consequence, it suffices to show that $\rho(\xi^\omega([a]))$ is α-substitutive. Let $M = (M_{i,j})_{i,j\in\Sigma}$ and $N = (N_{i,j})_{i,j\in\Delta}$ be the incidence matrices of g and ξ, respectively. By Definition 14 we have for all $a, b \in \Sigma$: $|\xi([a])|_{[b]} = |g(a)|_b$ and $|\xi(a)|_b = |\xi(a)|_{[b]} = 0$. Hence we obtain $N_{[b],[a]} = M_{b,a}$, $N_{b,a} = 0$ and $N_{[b],a} = 0$ for all $a, b \in \Sigma$. After changing the names (swapping a with $[a]$) in N, we obtain from Lemma 18 that N and M have the same non-zero eigenvalues, and thus the same dominant eigenvalue. ◻

4 Closure of Morphic Sequences under Transduction

In this section, we give a proof of the following theorem due to Dekking [4].

Theorem 20 (Transducts of morphic sequences are morphic). *If M is a transducer with input alphabet Σ and $x \in \Sigma^\omega$ is a morphic sequence, then $M(x)$ is morphic or finite.*

This proof will proceed by *annotating* entries in the original sequence x with information about what state the transducer is in upon reaching that entry. This allows us to construct a new morphism which produces the transduced sequence $M(x)$ as output. After proving this theorem, we will show that this process of annotation preserves α-substitutivity.

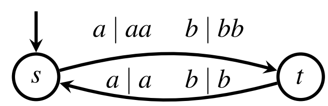

Fig. 2. A transducer that doubles every other letter.

4.1 Transducts of morphic sequences are morphic

We show in Lemma 27 that transducts of morphic sequences are morphic. In order to prove this, we also need several lemmas about transducers which are of independent interest. The approach here is adapted from a result in Allouche and Shallit [1]; it is attributed in that book to Dekking. We repeat it here partly for the convenience of the reader, but mostly because there are some details of the proof which are used in the analysis of the substitutivity property.

Definition 21 **(τ_w, $\Xi(w)$).** *Given a transducer $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ and a word $w \in \Sigma^*$, we define $\tau_w \in Q^Q$ to be $\tau_w(q) = \delta(q, w)$. Note that $\tau_{wv} = \tau_v \circ \tau_w$. Further, we define $\Xi : \Sigma^* \to (Q^Q)^\omega$ by $\Xi(w) = (\tau_w, \tau_{h(w)}, \tau_{h^2(w)}, \ldots, \tau_{h^n(w)}, \ldots)$.*

Next, we show that $\{\, \Xi(w) : w \in \Sigma^* \,\}$ is finite.

Lemma 22. *For any transducer M and any morphism $h : \Sigma \to \Sigma^*$, there are natural numbers $p \geq 1$ and $n \geq 0$ so that for all $w \in \Sigma^*$, $\tau_{h^i(w)} = \tau_{h^{i+p}(w)}$ for all $i \geq n$.*

Proof. Let $\Sigma = \{1, 2, \ldots, s\}$. Define $H : (Q^Q)^s \to (Q^Q)^s$ by $H(f_1, f_2, \ldots, f_s) = (f_{h(1)}, f_{h(2)}, \ldots, f_{h(s)})$. When we write $f_{h(i)}$ on the right, here is what we mean. Suppose that $h(i) = v_0 \cdots v_j$. Then $f_{h(i)}$ is short for the composition $f_{v_j} \circ f_{v_{j-1}} \circ \cdots \circ f_{v_1} \circ f_{v_0}$. Recall the notation τ_w from Definition 21; we thus have τ_i for the individual letters $i \in \Sigma$. Consider $T_0 = (\tau_1, \tau_2, \ldots, \tau_s)$. We define its *orbit* as the infinite sequence $(T_i)_{i \in \omega}$ of elements of $(Q^Q)^s$ given by $T_i = H^i(T_0) = H^i(\tau_1, \ldots \tau_s) = (\tau_{h^i(1)}, \ldots, \tau_{h^i(s)})$. Since each of the T_i belongs to the finite set $(Q^Q)^s$, the orbit of T_0 is eventually periodic. Let n be the preperiod length and p be the period length. The periodicity implies that $(*)$ $\tau_{h^i(j)} = \tau_{h^{i+p}(j)}$ for each $j \in \Sigma$ and for all $i \geq n$.

Let $w \in \Sigma^*$ and $i \geq n$. Since $w \in \Sigma^*$, we can write it as $w = \sigma_1 \sigma_2 \cdots \sigma_m$. We prove that $\tau_{h^i(w)} = \tau_{h^{i+p}(w)}$. Note that $\tau_{h^i(w)} = \tau_{h^i(\sigma_1 \cdots \sigma_m)} = \tau_{h^i(\sigma_1) h^i(\sigma_2) \cdots h^i(\sigma_m)} = \tau_{h^i(\sigma_n)} \circ \cdots \circ \tau_{h^i(\sigma_1)}$. We got this by breaking w into individual letters, then using the fact that h is a morphism, and finally using the fact that $\tau_{uv} = \tau_u \circ \tau_v$. Finally we know by $(*)$ that for individual letters, $\tau_{h^i(\sigma_j)} = \tau_{h^{i+p}(\sigma_j)}$. So $\tau_{h^i(w)} = \tau_{h^{i+p}(w)}$, as desired. $\square$

Definition 23 **($\Theta(w)$).** *Given a transducer M and a morphism h, we find p and n as in Lemma 22 just above and define $\Theta(w) = (\tau_w, \tau_{h(w)}, \ldots, \tau_{h^{n+p-1}(w)})$.*

Lemma 24. *(i) Given M and h, the set $A = \{\, \Theta(w) : w \in \Sigma^* \,\}$ is finite.*
(ii) If $\Theta(w) = \Theta(y)$, then $\Theta(h(w)) = \Theta(h(y))$.
(iii) If $\Theta(w) = \Theta(y)$, then for all $u \in \Sigma^*$, $\Theta(wu) = \Theta(yu)$.

Proof. Part (i) comes from the fact that each of the $n + p$ coordinates of $\Theta(w)$ comes from the finite set Q^Q. For (ii), we calculate:

$$\begin{aligned}\Theta(h(w)) &= (\tau_{h(w)}, \tau_{h^2(w)}, \tau_{h^3(w)}, \ldots, \tau_{h^{n+p}(w)}) = (\tau_{h(w)}, \tau_{h^2(w)}, \tau_{h^3(w)}, \ldots, \tau_{h^{n+p-1}(w)}, \tau_{h^n(w)}) \\ &= (\tau_{h(y)}, \tau_{h^2(y)}, \tau_{h^3(y)}, \ldots, \tau_{h^{n+p-1}(y)}, \tau_{h^n(y)}) = \Theta(h(y))\end{aligned}$$

using by Lemma 22 and since $\Theta(w) = \Theta(y)$. Part (iii) uses $\Theta(w) = \Theta(y)$ as follows:

$$\begin{aligned}\Theta(wu) &= (\tau_u \circ \tau_w, \tau_{h(u)} \circ \tau_{h(w)}, \tau_{h^2(u)} \circ \tau_{h^2(w)}, \ldots, \tau_{h^{n+p-1}(u)} \circ \tau_{h^{n+p-1}(w)}) \\ &= (\tau_u \circ \tau_y, \tau_{h(u)} \circ \tau_{h(y)}, \tau_{h^2(u)} \circ \tau_{h^2(y)}, \ldots, \tau_{h^{n+p-1}(u)} \circ \tau_{h^{n+p-1}(y)}) = \Theta(yu)\end{aligned}$$

$\square$

Definition 25 **($\overline{h}$).** *Given a transducer M and a morphism h, let A be as in Lemma 24(i). Define the morphism $\overline{h} : \Sigma \times A \to (\Sigma \times A)^*$ as follows. For for all $\sigma \in \Sigma$, whenever*

$h(\sigma) = s_1 s_2 s_3 \cdots s_\ell$, let $\overline{h}((\sigma, \Theta(w)))$ be defined as

$$(s_1, \Theta(hw)) \ \ (s_2, \Theta((hw)s_1)) \ \ (s_3, \Theta((hw)s_1 s_2)) \ \ \cdots \ \ (s_\ell, \Theta((hw)s_1 s_2 \cdots s_{\ell-1}))$$

By Lemma 24, $\overline{h}$ is well-defined. Notice that $|\overline{h}(\sigma, a)| = |h(\sigma)|$ for all σ.

Lemma 26. *For all $\sigma \in \Sigma$, all $w \in \Sigma^*$ and all $n \in \mathbb{N}$, if $h^n(\sigma) = s_1 s_2 \cdots s_\ell$, then*

$$\overline{h}^n((\sigma, \Theta(w))) \ = \ (s_1, \Theta(h^n w)) \ (s_2, \Theta((h^n w)s_1)) \ \cdots \ (s_\ell, \Theta((h^n w)s_1 s_2 \cdots s_{\ell-1})) \ .$$

In particular, for $1 \le i \le \ell$, the first component of the i^{th} term in $h^n(\sigma, \Theta(w))$ is s_i.

Proof. By induction on n. For $n = 0$, the claim is trivial. Assume that it holds for n. Let $h^n(\sigma) = s_1 s_2 \cdots s_\ell$, and for $1 \le i \le \ell$, let $h(s_i) = t^i_1 t^i_2 \cdots t^i_{k_i}$. Thus $h^{n+1}(\sigma) = h(s_1 s_2 \cdots s_\ell) = t^1_1 t^1_2 \cdots t^1_{k_1} t^2_1 t^2_2 \cdots t^2_{k_2} t^\ell_1 t^\ell_2 \cdots t^\ell_{k_\ell}$. Then:

$$\overline{h}(\overline{h}^n(\sigma, \Theta(w))) \ = \ \overline{h}(s_1, \Theta((h^n w))) \ \overline{h}(s_2, \Theta((h^n w)s_1)) \ \cdots \ \overline{h}(s_\ell, \Theta((h^n w)s_1 s_2 \cdots s_{\ell-1}))$$

For $1 \le i \le \ell$, we have

$$\begin{aligned}
&\overline{h}(s_i, \Theta((h^n w)s_1 \cdots s_{i-1})) \\
&= (t^i_1, \Theta((hh^n w)h(s_1 \cdots s_{i-1}))) \quad (t^i_2, \Theta((hh^n w)h(s_1 \cdots s_{i-1})t^i_1)) \\
&\qquad \cdots \quad (t^i_{k_i}, \Theta(hh^n w)h(s_1 \cdots s_{i-1})t^i_1 t^i_2 \cdots t^i_{k_i-1})) \\
&= (t^i_1, \Theta((h^{n+1} w)t^1_1 t^1_2 \cdots t^1_{k_1} \cdots t^{i-1}_1 t^{i-1}_2 \cdots t^{i-1}_{k_{i-1}})) \quad (t^i_2, \Theta((h^{n+1} w)t^1_1 t^1_2 \\
&\qquad \cdots \quad t^1_{k_1} \cdots t^{i-1}_1 t^{i-1}_2 \cdots t^{i-1}_{k_{i-1}} t^i_1)) \\
&\qquad \cdots \quad (t^i_{k_i}, \Theta((h^{n+1} w)t^1_1 t^1_2 \cdots t^1_{k_i} \cdots t^{i-1}_1 t^{i-1}_2 \cdots t^{i-1}_{k_{i-1}} t^i_1 \cdots t^i_{k_i-1}))
\end{aligned}$$

Concatenating the sequences $\overline{h}(s_i, \Theta((h^n w)s_1 \cdots s_{i-1}))$ for $i = 1, \ldots, \ell$ completes our induction step. ◻

Lemma 27. *Let $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ be a transducer, let h be a morphism prolongable on the letter x_1, and write $h^\omega(x_1)$ as $x = x_1 x_2 x_3 \cdots x_n \cdots$. Let Θ be from Definition 23. Using this, let A be from Lemma 24(i), and $\overline{h}$ from Definition 25. Then*

(i) $\overline{h}$ is prolongable on $(x_1, \Theta(\epsilon))$.
(ii) Let $c : \Sigma \times A \to \Sigma \times Q$ be the coding $c(\sigma, \Theta(w)) = (\sigma, \tau_w(q_0))$. Then c is well-defined.
(iii) The image under c of $\overline{h}^\omega((x_1, \Theta(\epsilon))$ is

$$z \ = \ (x_1, \delta(q_0, \epsilon)) \ (x_2, \delta(q_0, x_1)) \ \cdots \ (x_n, \delta(q_0, x_1 x_2 \cdots x_{n-1})) \ \cdots \qquad (1)$$

This sequence z is morphic in the alphabet $\Sigma \times Q$.

Proof. For (i), write $h(x_1)$ as $x_1 x_2 \cdots x_\ell$. Using the fact that $h^i(\epsilon) = \epsilon$ for all i, we see that $\overline{h}((x_1, \Theta(\epsilon))) = (x_1, \Theta(\epsilon)) \quad (x_2, \Theta(x_1)) \quad \cdots \quad (x_\ell, \Theta(x_1, \ldots, x_{\ell-1}))$. This verifies the prolongability. For (ii): if $\Theta(w) = \Theta(u)$, then τ_w and τ_u are the first component of $\Theta(w)$ and are thus equal. We turn to (iii). Taking $w = \epsilon$ in Lemma 26 shows that $\overline{h}^\omega((x_1, \Theta(\epsilon))$ is

$$(x_1, \Theta(\epsilon)) \quad (x_2, \Theta(x_1)) \quad (x_3, \Theta(x_1 x_2)) \quad \cdots \quad (x_m, \Theta(x_1 x_2 \cdots x_{m-1})) \quad \cdots \ .$$

The image of this sequence under the coding c is

$$(x_1, \tau_\epsilon(q_0)) \quad (x_2, \tau_{x_1}(q_0)) \quad (x_3, \tau_{x_1x_2}(q_0)) \quad \cdots \quad (x_m, \tau_{x_1x_2\cdots x_{m-1}}(q_0)) \quad \cdots .$$

In view of the τ functions' definition (Def. 21), we obtain z in (1). By definition, z is morphic. $\square$

This is most of the work required to prove Theorem 20, the main result of this section.

Proof (Theorem 20). Since x is morphic there is a morphism $h : \Sigma' \to (\Sigma')^*$, a coding $c : \Sigma' \to \Sigma$, and an initial letter $x_1 \in \Sigma'$ so that $x = c(h^\omega(x_1))$. We are to show that $M(c(h^\omega(x_1)))$ is morphic. Since c is computable by a transducer, we have $x = (M \circ c)(h^\omega(x_1))$, where $\circ$ is the wreath product of transducers. It is thus sufficient to show that given a transducer M, the sequence $M(h^\omega(x_1))$ is morphic.

By Lemma 27, the sequence $z = (x_1, \delta(q_0, \epsilon))\ (x_2, \delta(q_0, x_1))\ (x_3, \delta(q_0, x_1x_2))\ \cdots$ is morphic. The output function of M is a morphism $\lambda : \Sigma \times Q \to \Delta^*$. By Corollary 16, $\lambda(z)$ is morphic or finite. But $\lambda(z)$ is exactly $M(x)$. $\square$

4.2 Substitutivity of transducts

We are also interested in analyzing the α-substitutivity of transducts. We claim that if a sequence x is α-substitutive, then $M(x)$ is also α-substitutive for all M.

As a first step, we show that annotating a morphism does not change α-substitutivity.

Definition 28. *Let Σ be an alphabet, A any set and $w = (b_1, a_1)\ (b_2, a_2)\ \ldots\ (b_k, a_k) \in (\Sigma \times A)^*$ be a word. We call A the* set of annotations. *We write $\lfloor w \rfloor$ for the word $b_1 b_2 \ldots b_k$, that is, the word obtained by* dropping the annotations.

A morphism $\overline{h} : (\Sigma \times A) \to (\Sigma \times A)^*$ is an annotation *of $h : \Sigma \to \Sigma^*$ if $h(b) = \lfloor \overline{h}(b, a) \rfloor$ for all $b \in \Sigma$, $a \in A$.*

Note that the morphism $\overline{h}$ from Definition 25 is an annotation of h in this sense. Then from the following proposition it follows that if x is α-substitutive, then the sequence z in Lemma 27 is also α-substitutive.

Proposition 29. *If $x = h^\omega(\sigma)$ is an α-substitutive morphic sequence with morphism $h : \Sigma \to \Sigma^*$ and A is any set of annotations, then any annotated morphism $\overline{h} : \Sigma \times A \to (\Sigma \times A)^*$ also has an infinite fixpoint $\overline{h}^\omega((\sigma, a))$ which is also α-substitutive.*

The proof of this proposition is in two lemmas: first that the eigenvalues of the morphism are preserved by the annotation process, and second that if α is the dominant eigenvalue for h, then no greater eigenvalues are introduced for $\overline{h}$.

Lemma 30. *All eigenvalues for h are also eigenvalues for any annotated version $\overline{h}$ of h.*

Proof. Let $M = (m_{i,j})_{i,j \in \Sigma}$ be the incidence matrix of h. Order the elements of the annotated alphabet $\Sigma \times A$ lexicographically. Then the incidence matrix of $\overline{h}$, call it $N = (n_{i,j})_{i,j \in \Sigma \times A}$, can be thought of as a block matrix where the blocks have size $|A| \times |A|$ and there are $|\Sigma| \times |\Sigma|$ such blocks in N. Note that by the definition of annotation, the row sum in each row of the (a, b) block of N is $m_{a,b}$. To simplify the notation, for the rest of this proof we write J for $|\Sigma|$ and K for $|A|$. Suppose $v = (v_1, v_2, \ldots, v_J)$ is a column eigenvector for M with eigenvalue α. Consider $\overline{v} = (v_1, \ldots, v_1, v_2, \ldots, v_2, \ldots, v_n, \ldots v_n)$. This is a "block vector": the first K entries are v_1, the second K entries are v_2, and so on, for a total of $K \cdot J$ entries. We claim that $\overline{v}$ is a column eigenvector for N with eigenvalue α.

Consider the product of row k of N with $\overline{v}$. This is $\sum_{j=1}^{K \cdot J} n_{k,j} \overline{v}_j = \sum_{b=1}^{J} v_b \cdot (\sum_{j=1}^{K} n_{k,Kb+j})$. Now $k = Ka + r$. So $\sum_{j=1}^{K} n_{k,Kb+j}$ is the row sum of the (a, b) block of N and hence is $m_{a,b}$. Therefore, row k of N times $\overline{v}$ is $\sum_{b=1}^{J} v_b m_{a,b} = \alpha v_a$, since v is an eigenvector of M. Finally we note that the kth entry of $\overline{v}$ is v_a by its definition. Hence multiplying $\overline{v}$ by N multiplies the kth entry of $\overline{v}$ by α for all k.

We have shown that $\overline{v}$ is a column eigenvector of N with eigenvalue α, so the (column) eigenvalues of M are all present in N. However, since a matrix and its transpose have the same eigenvalues, the (column) qualification on the eigenvalues is unnecessary. □

If $\overline{h}$ is an annotation of h, then we have

$$|h(b)|_{b'} = \sum_{a' \in A} |\,\overline{h}((b, a))\,|_{(b', a')} \qquad \text{for all } b, b' \in \Sigma \text{ and } a \in A \tag{2}$$

Lemma 31. *Let $h, \overline{h}$ be morphisms such that $\overline{h} : (\Sigma \times A) \to (\Sigma \times A)^*$ is an annotation of $h : \Sigma \to \Sigma^*$. Then every eigenvalue of $\overline{h}$ with a non-negative eigenvector is also an eigenvalue for h.*

Proof. Let $M = (m_{i,j})_{i,j \in \Sigma}$ be the incidence matrix of h and $N = (n_{i,j})_{i,j \in \Sigma \times A}$ be the incidence matrix of $\overline{h}$. Let r be an eigenvalue of N with corresponding eigenvector $v = (v_{(b,a)})_{(b,a) \in \Sigma \times A}$, that is, $Nv = rv$ and $v \neq 0$. We define a vector $w = (w_b)_{b \in \Sigma}$ as follows: $w_b = \sum_{a \in A} v_{(b,a)}$. We show that $Mw = rw$. Let $b' \in \Sigma$, then:

$$\begin{aligned}
(Mw)_{b'} &= \sum_{b \in \Sigma} M_{b',b} w_b = \sum_{b \in \Sigma} \left(M_{b',b} \sum_{a \in A} v_{(b,a)} \right) = \sum_{b \in \Sigma} \sum_{a \in A} M_{b',b} v_{(b,a)} \\
&\overset{\text{by (2)}}{=} \sum_{b \in \Sigma} \sum_{a \in A} \left(\sum_{a' \in A} N_{(b',a'),(b,a)} \right) v_{(b,a)} = \sum_{a' \in A} \sum_{b \in \Sigma} \sum_{a \in A} N_{(b',a'),(b,a)} v_{(b,a)} \\
&\overset{Nv=rv}{=} \sum_{a' \in A} r v_{(b',a')} = r \sum_{a' \in A} v_{(b',a')} = r w_{b'}
\end{aligned}$$

Hence $Mw = rw$. If $w \neq 0$ it follows that r is an eigenvalue of M. Note that if v is non-negative, then $w \neq 0$. This proves the claim. □

Corollary 32. *Let $h, \overline{h}$ be morphisms such that $\overline{h} : (\Sigma \times A) \to (\Sigma \times A)^*$ is an annotation of $h : \Sigma \to \Sigma^*$. Then the dominant eigenvalue for h coincides with the dominant eigenvalue for $\overline{h}$.*

Proof. By Lemma 30 every eigenvalue of h is an eigenvalue of $\overline{h}$. Thus the dominant eigenvalue of $\overline{h}$ is greater or equal to that of h. By Theorem 4, the dominant eigenvalue of a non-negative matrix is a real number $\alpha > 1$ and its corresponding eigenvector is non-negative. By Lemma 30, every eigenvalue of $\overline{h}$ with a non-negative eigenvector is also an eigenvalue of h. Thus the dominant eigenvalue of h is also greater or equal to that of $\overline{h}$. Hence the dominant eigenvalues of h and $\overline{h}$ must be equal. $\square$

Theorem 33. *Let α and β be multiplicatively independent real numbers. If v is a α-substitutive sequence and w is an β-substitutive sequence, then v and w have no common non-erasing transducts except for the ultimately periodic sequences.*

Proof. Let h_v and h_w be morphisms whose fixed points are v and w, respectively. By the proof of Theorem 20, x is a morphic image of an annotation $\overline{h}_v$ of h_v, and also of an annotation $\overline{h}_w$ of h_w. The morphisms must be non-erasing, by the assumption in this theorem. By Corollary 32 and Theorem 19, x is both α- and β-substitutive. By Durand's Theorem 6, x is eventually periodic. $\square$

5 Conclusion

We have re-proven some of the central results in the area of morphic sequences, the closure of the morphic sequences under morphic images and transduction. However, the main results in this paper come from the eigenvalue analyses which followed our proofs in Sections 3 and 4. These are some of the only results known to us which enable one to prove negative results on the transducibility relation $\trianglelefteq$. One such result is in Theorem 33; this is perhaps the culmination of this paper. The next step in this line of work is to weaken the hypothesis in some of results that the transducers be *non-erasing*.

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